

FD methods for diffusion (II)

The transient time-dependent two-dimensional heat equation in (x,z) space reads (see course script “basic equations”)

$$\rho c_p \frac{\partial T}{\partial t} = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) + Q \quad (1)$$

where ρ is the density, c_p heat capacity, k thermal conductivity and Q is the radiogenic heat production.

The simplest way to discretize equation (1) on a domain, e.g. a box with width L and height H , is again the explicit FTCS scheme

$$\frac{T_{i,j}^{n+1} - T_{i,j}^n}{\Delta t} = \kappa \left(\frac{T_{i,j+1}^n - 2T_{i,j}^n + T_{i,j-1}^n}{\Delta x^2} + \frac{T_{i+1,j}^n - 2T_{i,j}^n + T_{i-1,j}^n}{\Delta z^2} \right) + \frac{Q_{i,j}^n}{\rho c_p}$$

There are now two indices for space dependency: i for z -direction (depth) and j for x -direction (width). Rearranging gives

$$T_{i,j}^{n+1} = T_{i,j}^n + \beta_x (T_{i,j+1}^n - 2T_{i,j}^n + T_{i,j-1}^n) + \beta_z (T_{i+1,j}^n - 2T_{i,j}^n + T_{i-1,j}^n) + \frac{Q_{i,j}^n \Delta t}{\rho c_p}$$

where

$$\beta_x = \frac{\kappa \Delta t}{\Delta x^2} \quad \text{and} \quad \beta_z = \frac{\kappa \Delta t}{\Delta z^2} \quad .$$

Boundary conditions can be set the usual way. A constant temperature (Dirichlet BC) T_{left} on the left hand side of the domain (at $j = 1$), for example, is given by

$$T_{i,j=1} = T_{left} \quad \forall i \quad .$$

A constant flux (Neumann BC) on the same boundary may be set through so-called “fictitious boundary” points (not discussed here).

The major disadvantage of fully explicit schemes is, of course, that they are only stable if

$$\frac{2 \kappa \Delta t}{\min(\Delta x^2, \Delta z^2)} \leq 1 \quad .$$

If we employ instead a fully implicit and unconditionally stable discretization scheme as for the 1D case, the above expression can be written as (in complete analogy)

$$\frac{T_{i,j}^{n+1} - T_{i,j}^n}{\Delta t} = \kappa \left(\frac{T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j-1}^{n+1}}{\Delta x^2} + \frac{T_{i+1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i-1,j}^{n+1}}{\Delta z^2} \right) + \frac{Q_{i,j}^n}{\rho c_p}$$

Rearranging terms with $n + 1$ on the left and terms with n on the right hand side gives (without the term $Q_{i,j}^n \Delta t / \rho c_p$)

$$-\beta_z T_{i+1,j}^{n+1} - \beta_x T_{i,j+1}^{n+1} + (1 + 2\beta_z + 2\beta_x) T_{i,j}^{n+1} - \beta_z T_{i-1,j}^{n+1} - \beta_x T_{i,j-1}^{n+1} = T_{i,j}^n$$

As in the 1D case, we have to put these equations in a matrix **A** and vector **rhs** and use Matlab's "`x = A\rhs`" to solve for T^{n+1} .

From a practical point of view, however, this is a bit more complicated than the 1D case, since we have to deal with "book-keeping" issues, i.e. the mapping of $T_{i,j}$ to the entries of a temperature vector $T(k)$ – as opposed to the more intuitive matrix $T(i,j)$ we were allowed to in the case of an explicit scheme.

If a 2D temperature field is to be solved for with an equivalent vector T , the nodes have to be numbered continuously !

The derivative versus the x -direction is then e.g. ($n_x = 7, n_z = 5$)

$$\frac{\partial^2 T}{\partial x^2} \Big|_{i=3, j=4} = \frac{\partial^2 T_{18}}{\partial x^2} = \frac{1}{\Delta x^2} (T_{19} - 2T_{18} + T_{17})$$

and the derivative versus z -direction is given by

$$\frac{\partial^2 T}{\partial z^2} \Big|_{i=3, j=4} = \frac{\partial^2 T_{18}}{\partial z^2} = \frac{1}{\Delta z^2} (T_{25} - 2T_{18} + T_{11})$$

If n_x are the number of grid points in x -direction and n_z the number of points in z -direction, we can write these expressions in the general form

$$\frac{\partial^2 T}{\partial x^2} \Big|_{i,j} = \frac{1}{\Delta x^2} (T_{(i-1)n_x+j+1} - 2T_{(i-1)n_x+j} + T_{(i-1)n_x+j-1})$$

and

$$\frac{\partial^2 T}{\partial z^2} \Big|_{i,j} = \frac{1}{\Delta z^2} (T_{i \cdot n_x+j} - 2T_{(i-1)n_x+j} + T_{(i-2)n_x+j})$$

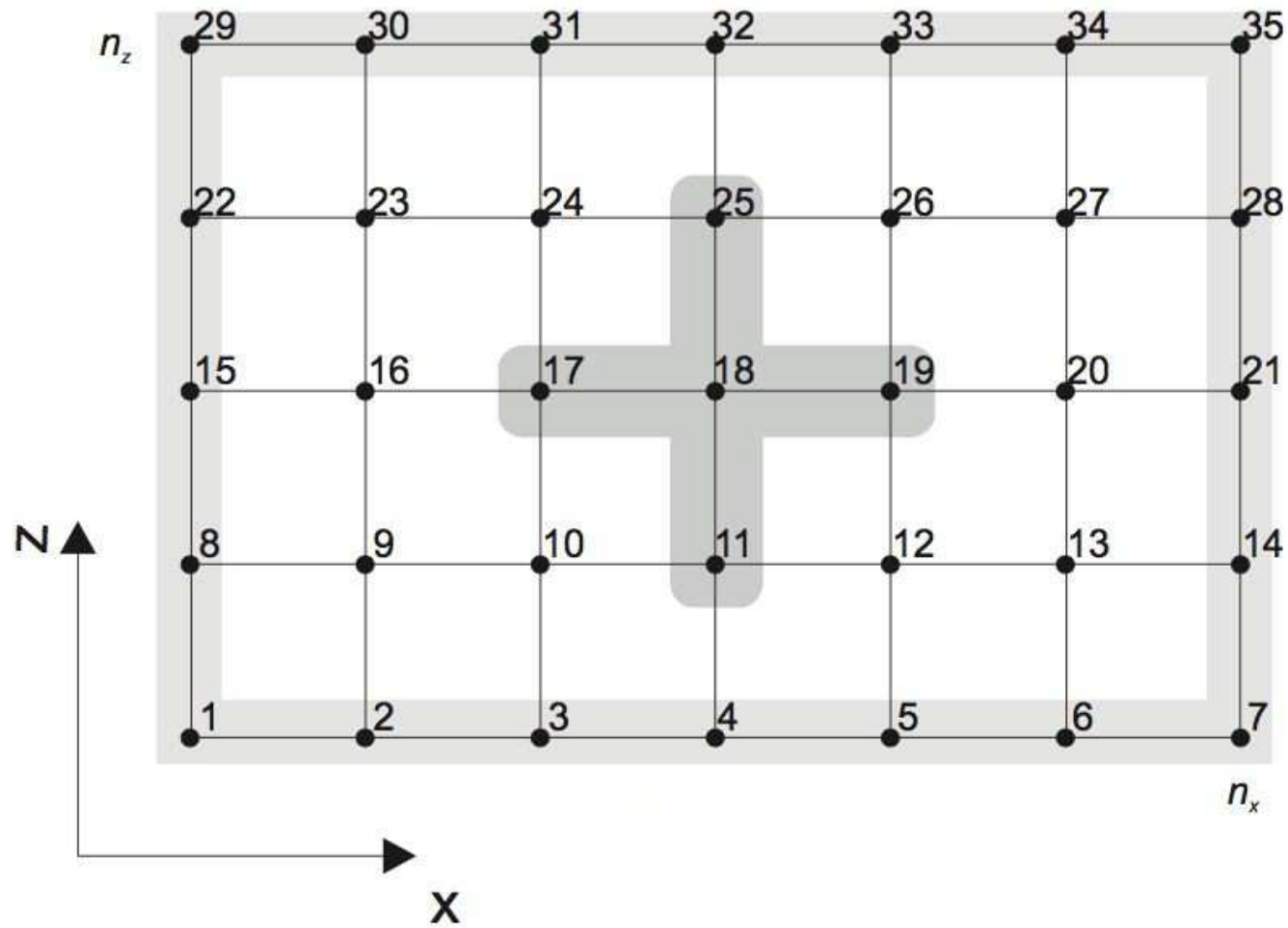


Figure: Numbering scheme for a 2D grid with $n_x = 7$ and $n_z = 5$

Exercise

The problem to be considered is that of the thermal structure of a lithosphere of 100 km thickness, with an initial thermal gradient of 13 K/km. Suddenly a plume with $T = 1500$ degree Celsius impinges at the bottom of the lithosphere.

Fill in the question marks in the script `heat2D_template.m` at the course website. Employ Dirichlet boundary conditions on edges of the domain. Ignore the effects of radioactive heat.

What happens with the thermal structure of the lithosphere ? –
A related (structural geology) problem is that of the cooling of batholiths (like the ones in the Sierra Nevada).