CHAPTER 3

Representation of Seismic Sources

Seismic waves are set up by winds, ocean waves, meteorite impacts, rocket launchings, and atmospheric explosions—even by people walking around in the vicinity of seismometers. These, however, are examples of sources external to the solid Earth, and they can usually be analyzed within the simple framework of time-varying tractions applied to the Earth's surface. Other sources that, for many practical purposes, are also external, include volcanic eruptions, vented explosions, and spalling (free fall of a surface layer thrown upward by an underground explosion). For internal sources, such as earthquakes and underground explosions, the analytical framework is more difficult to develop, because the equations governing elastic motion (2.17)–(2.18) do not hold throughout the solid Earth. This chapter is about internal sources, and we shall distinguish two different categories: faulting sources and volume sources.

A faulting source is an event associated with an internal surface, such as slip across a fracture plane. A volume source is an event associated with an internal volume, such as a sudden (explosive) expansion throughout a volumetric source region. We shall find that a unified treatment of both source types is possible, the common link being the concept of an internal surface across which discontinuities can occur in displacement (for the faulting source) or in strain (for the volume source).

The mathematical description of internal seismic sources has classically been pursued along two different lines: first, in terms of a body force applied to certain elements of the medium containing the source; and second, by some discontinuity in displacement or strain (e.g., across a rupturing fault surface or across the surface of a volume source). The second approach can usefully be incorporated into the first if we can find body-force equivalents to discontinuities in displacement and strain. We begin our analysis by developing body-force equivalents in some detail for simple shearing across a fault surface, showing that radically different systems of forces can be equivalent to exactly the same displacement discontinuity. We then develop the general theory for faulting sources, following Burridge and Knopoff (1964), and finally we outline the theory for a volume source.

The motions recorded in a seismogram are a result both of propagation effects and of source effects. Thus a major reason for seeking a better understanding of the source mechanism has been to isolate the propagation effects, since these bear information on the Earth's internal structure. Since the pioneering work of Sykes (1967), earthquake source mechanisms have been studied to chart the motions of tectonic plates. Source theory can elucidate physical processes such as those taking place in volcanoes. It continues to be developed with a view to predicting earthquake hazards at engineering sites, on the basis of geological and geophysical data on the properties of nearby faults and the distribution of regional stresses.

3.1 Representation Theorems for an Internal Surface; Body-Force Equivalents for Discontinuities in Traction and Displacement

The representation theorems obtained in Chapter 2 can be a powerful aid in seismic source theory if the surface S is chosen to include two adjacent surfaces internal to the volume V. The motivation here comes from the work of H. F. Reid, whose study of the San Andreas fault before and after the 1906 San Francisco earthquake led to general recognition that earthquake motion is due to waves radiated from spontaneous slippage on active geological faults. We shall discuss this source mechanism in more detail in Sections 3.2 and 3.3, and the dynamical processes involved (and other source mechanisms) in Chapter 11. Our present concern is simply to show how the process of slip on a buried fault, and the waves radiated from it, can naturally be analyzed by our representation theorems.

For applications of (2.41)–(2.43), we shall take the surface of V to consist of an external surface labeled S (see Fig. 3.1) and two adjacent internal surfaces, labeled Σ^+ and Σ^-, which are opposite faces of the fault. If slip occurs across Σ, then the displacement field is discontinuous there and the equation of motion is no longer satisfied throughout the interior of S. However, it *is* satisfied throughout the "interior" of the surface $S + \Sigma^+ + \Sigma^-$, and to this we can apply our previous representation results.

The surface S is no longer of direct interest (it may be the surface of the Earth), and we shall assume that both **u** and **G** satisfy the same homogeneous boundary conditions on

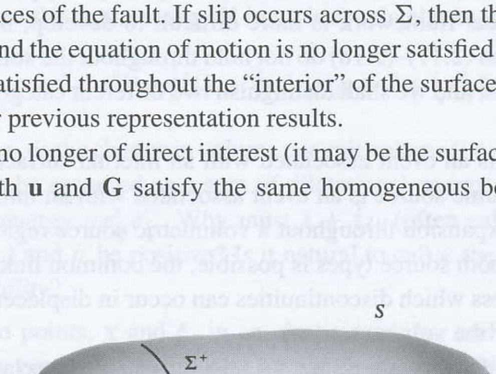

FIGURE **3.1**
A finite elastic body, with volume V and external surface S, and an internal surface Σ (modeling a buried fault) across which discontinuities may arise. That is, displacements on the Σ^- side of Σ may differ from displacements on the Σ^+ side of Σ. The normal to Σ is ν (pointing from Σ^- to Σ^+), and the displacement discontinuity is denoted by $[\mathbf{u}(\xi, \tau)]$ for ξ on Σ, with square brackets referring to the difference $\mathbf{u}(\xi, \tau)|_{\Sigma^+} - \mathbf{u}(\xi, \tau)|_{\Sigma^-}$. In general, a similar difference may be formed for the tractions (due to external applied forces on Σ), but for spontaneous rupture the tractions must be continuous, and then $[\mathbf{T}(\mathbf{u}, \nu)] = \mathbf{0}$.

S—though not on Σ (see below). Then from (2.41), using (2.39) and renaming some variables and indices,

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\tau \iiint_V f_p(\eta, \tau) G_{np}(\mathbf{x}, t - \tau; \eta, 0)\, dV(\eta)$$
$$+ \int_{-\infty}^{\infty} d\tau \iint_\Sigma \Big\{ \big[u_i(\xi, \tau) c_{ijpq} \nu_j \partial G_{np}(\mathbf{x}, t - \tau; \xi, 0)/\partial \xi_q \big]$$
$$- \big[G_{np}(\mathbf{x}, t - \tau; \xi, 0) T_p(\mathbf{u}(\xi, \tau), \nu) \big] \Big\}\, d\Sigma. \tag{3.1}$$

This formula uses η as the general position within V, and ξ as the general position on Σ. Square brackets are used for the difference between values on Σ^+ and Σ^- (see caption for Fig. 3.1).

As yet, nothing has been assumed for the boundary conditions on Σ. Although the choice for $\mathbf{u}$ must conform to actual properties of displacement and traction across a rupturing fault surface, the choice for $\mathbf{G}$ can be made in any fashion that turns out to be useful. Thus, for $\mathbf{u}$, the slip on a fault leads to a nonzero value for $[\mathbf{u}]$, but the continuity of traction (see the proof of (2.7), and Problem 2.8) implies $\big[\mathbf{T}(\mathbf{u}, \nu)\big] = \mathbf{0}$. The simplest and most commonly used way to establish a defining property of $\mathbf{G}$ on Σ is to take Σ as an artificial surface across which $\mathbf{G}$ and its derivatives are continuous, so that $\mathbf{G}$ satisfies the equation of motion (2.36) even on Σ. This is by far the easiest Green function to compute for the volume V, and (in the absence of body forces for $\mathbf{u}$) it gives the representation

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\tau \iint_\Sigma \big[u_i(\xi, \tau) \big]\, c_{ijpq} \nu_j \frac{\partial}{\partial \xi_q} G_{np}(\mathbf{x}, t - \tau; \xi, 0)\, d\Sigma. \tag{3.2}$$

It is not surprising that displacement on the fault is enough to determine displacement everywhere: this feature of (3.2) might have been expected from the uniqueness theorem. Nevertheless, it *is* at first sight surprising that no boundary conditions on Σ are needed for the Green function that describes waves propagating from the source. One might expect that motions occurring on the fault would set up waves that are themselves diffracted in some fashion by the fault surface. But although this interaction complicates the determination of the slip function $[\mathbf{u}(\xi, \tau)]$, it does not enter into the determination of the Green function used in (3.2), and many seismologists have used this formula to compute the motions set up by some assumed model of the slip function. We shall describe examples of such integrations in Chapter 10.

3.1.1 BODY-FORCE EQUIVALENTS

The earthquake model we have just described does not directly involve any body forces, though the representation (3.2) does give displacement at $(\mathbf{x}, t)$ as an integral over contributing Green functions, each of which is the same as if it had been set up by a body force. Thus there must be some sense in which an active fault surface can be regarded as a surface distribution of body forces.

To determine this body-force equivalent, we start with (3.1) and assume still that Σ is transparent to $\mathbf{G}$. Making no assumptions about $[\mathbf{u}]$ and $[\mathbf{T}(\mathbf{u}, \mathbf{n})]$ across Σ (so that sources of traction are also allowed), we find

$$u_n(x,t) = \int_{-\infty}^{\infty} d\tau \iiint_V f_p(\eta,\tau) G_{np}(\mathbf{x}, t-\tau; \eta, 0)\, dV(\eta)$$
$$+ \int_{-\infty}^{\infty} d\tau \iint_\Sigma \Big\{ [u_i(\xi,\tau)]\, c_{ijpq} \nu_j G_{np,q}(\mathbf{x}, t-\tau; \xi, 0)$$
$$- \Big[T_p(\mathbf{u}(\xi,\tau), \nu)\Big] G_{np}(\mathbf{x}, t-\tau; \xi, 0) \Big\}\, d\Sigma(\xi). \tag{3.3}$$

The discontinuities on Σ can be localized within V by using the delta function $\delta(\eta - \xi)$. For example, $[\mathbf{T}]\, d\Sigma(\xi)$ has the dimensions of force, and its body-force distribution (i.e., force/unit volume) is $[\mathbf{T}]\, \delta(\eta - \xi) d\Sigma$ as η varies throughout V. The traction discontinuity in (3.3) therefore contributes the displacement

$$\int_{-\infty}^{\infty} d\tau \iiint_V \Big\{ -\iint_\Sigma \Big[T_p(\mathbf{u}(\xi,\tau), \nu)\Big]\, \delta(\eta - \xi)\, d\Sigma \Big\} G_{np}(\mathbf{x}, t-\tau; \eta, 0)\, dV.$$

Since this expression has precisely the form of a body-force contribution (see the first term in the right-hand side of (3.3)), the body-force equivalent of a traction discontinuity on Σ is given by $\mathbf{f}^{[\mathbf{T}]}$, where

$$\mathbf{f}^{[\mathbf{T}]}(\eta,\tau) = -\iint_\Sigma [\mathbf{T}(\mathbf{u}(\xi,\tau), \nu)]\, \delta(\eta - \xi)\, d\Sigma(\xi). \tag{3.4}$$

The displacement discontinuity is harder to interpret, displacement being less simply related to force than is traction. We use the delta-function derivative $\partial\delta(\eta - \xi)/\partial\eta_q$ to localize points of Σ within V. This function has the property

$$\frac{\partial}{\partial \xi_q} G_{np}(\mathbf{x}, t-\tau; \xi, 0) = -\iiint_V \frac{\partial}{\partial \eta_q} \delta(\eta - \xi) G_{np}(\mathbf{x}, t-\tau; \eta, 0)\, dV(\eta),$$

so that the displacement discontinuity in (3.3) contributes the displacement

$$\int_{-\infty}^{\infty} d\tau \iiint_V \Big\{ -\iint_\Sigma [u_i(\xi,\tau)]\, c_{ijpq} \nu_j \frac{\partial}{\partial \eta_q} \delta(\eta - \xi)\, d\Sigma \Big\} G_{np}(\mathbf{x}, t-\tau; \eta, 0)\, dV$$

at position $\mathbf{x}$ and time t. The body-force equivalent $\mathbf{f}^{[\mathbf{u}]}$ of a displacement discontinuity on Σ can now be recognized from this expression as

$$f_p^{[\mathbf{u}]}(\eta,\tau) = -\iint_\Sigma [u_i(\xi,\tau)]\, c_{ijpq} \nu_j \frac{\partial}{\partial \eta_q} \delta(\eta - \xi)\, d\Sigma. \tag{3.5}$$

BOX **3.1**

On the use of effective slip and effective elastic moduli in the source region

We are using the words "fault plane" and "fault surface," symbolized by Σ, as mathematical entities that have no thickness. Yet there are many places in the world where Earth scientists have direct access to fault regions, and one often finds there a zone of crushed and deformed rock, perhaps several meters thick, so that geologists often speak of "fault gouge" and a "fault zone." What, then, is meant by our claim that body-force equivalents depend only on elastic moduli at the fault surface?

The fault zone itself may be as wide as 200 meters, which for most but not all purposes is far less than the wavelengths of detectable seismic radiation, in which case it is the displacement change across the whole fault zone that is the apparent displacement discontinuity, initiating waves which propagate out of the source region. Therefore, in almost all practical cases, the elastic moduli for equations (3.2), (3.3), and (3.5) are the constants appropriate for the competent (unaltered) rock adjoining the fault zone. Exceptions may arise with fault zone effects that may be significant for seismic wave excitation at frequencies of interest to strong motion seismology (Aki, 1996).

Although the integrand here involves 27 terms (summation over i, j, q), which are different for each p, we shall find important examples in which only two or three terms are nonzero. The body-force equivalents (3.4) and (3.5) hold for a general inhomogeneous anisotropic medium, and they are remarkable in their dependence on properties of the elastic medium only at the fault surface itself.

Since faulting within the volume V is an internal process, the total momentum and total angular momentum must be conserved. It follows that the total force due to $\mathbf{f}^{[\mathbf{u}]}$, and the total moment of $\mathbf{f}^{[\mathbf{u}]}$ about any fixed point, must be zero. Thus

$$\iiint_V \mathbf{f}^{[\mathbf{u}]}(\boldsymbol{\eta}, \tau)\, dV(\boldsymbol{\eta}) = \mathbf{0} \qquad \text{for all } \tau, \tag{3.6}$$

and

$$\iiint_V (\boldsymbol{\eta} - \boldsymbol{\eta}_0) \times \mathbf{f}^{[\mathbf{u}]}(\boldsymbol{\eta}, \tau)\, dV(\boldsymbol{\eta}) = \mathbf{0} \qquad \text{for all } \tau \text{ and any fixed } \boldsymbol{\eta}_0. \tag{3.7}$$

To verify (3.6), note that the p-component of the left-hand side is $-\iint_\Sigma [u_i]\, c_{ijpq} \nu_j \left\{ \iiint_V \partial\delta(\boldsymbol{\eta} - \boldsymbol{\xi})/\partial\eta_q dV \right\} d\Sigma(\boldsymbol{\xi})$. The volume integral here is $\iint_S \delta(\boldsymbol{\eta} - \boldsymbol{\xi}) n_q\, dS(\boldsymbol{\eta})$, which vanishes because $\boldsymbol{\eta}$ on S can never equal $\boldsymbol{\xi}$ (S and Σ having no common point).

To verify (3.7), write the m-component of the left-hand side as

$$\iiint_V \varepsilon_{mnp}(\eta_n - \eta_{0n}) f_p^{[\mathbf{u}]} dV$$

$$= -\iint_\Sigma c_{ijpq}\nu_j [u_i] \left\{ \iiint_V \varepsilon_{mnp}(\eta_n - \eta_{0n}) \frac{\partial}{\partial \eta_q} \delta(\eta - \xi)\, dV \right\} d\Sigma \quad \text{(from (3.5))}$$

$$= + \iint_\Sigma \varepsilon_{mqp} c_{ijpq} \nu_j [u_i]\, d\Sigma \qquad \left(\text{using } \frac{\partial}{\partial \eta_q}(\eta_n - \eta_{0n}) = \delta_{nq}\right)$$

$$= 0 \qquad (\text{using the symmetry } c_{ijpq} = c_{ijqp}).$$

As a simple example of a body force that is equivalent to a field discontinuity, consider the case of a body force applied at just one point, and in a particular direction (e.g., the body force for a Green function, given by (2.4)). This can instead be regarded as a discontinuity in a component of stress. To obtain the equivalence, take x_3 as the depth direction and consider a vertical point force, with magnitude F, applied at $(0, 0, h)$ and time $\tau = 0$ and held steady. Then

$$\mathbf{f}(\eta, \tau) = (0, 0, F)\delta(\eta_1)\delta(\eta_2)\delta(\eta_3 - h)H(\tau). \tag{3.8}$$

This source can instead be regarded as a discontinuity in traction across one point of the plane $\xi_3 = h$, with

$$\left[\mathbf{T}(\xi, \tau)\right]_{\xi=(\xi_1,\xi_2,h^-)}^{\xi=(\xi_1,\xi_2,h^+)} = -(0, 0, F)\delta(\xi_1)\delta(\xi_2)H(\tau), \tag{3.9}$$

i.e., τ_{13}, τ_{23} are continuous, and the jump is in τ_{33}. The equivalence of (3.8) and (3.9) can be shown by a straightforward application of (3.4).

The most important example of a body-force equivalent in seismology is found in shear faulting, and we next take up this subject in some detail.

3.2 A Simple Example of Slip on a Buried Fault

The seismic waves set up by fault slip are the same as those set up by a distribution on the fault of certain forces with canceling moment. The distribution (for given fault slip) is not unique, but in an isotropic medium it can always be chosen as a surface distribution of double couples. This conclusion was unexpected, in view of arguments used in a long-lasting debate on the question of whether earthquakes should be modeled by a single couple or by a double couple. Those who advocated the single-couple theory did believe that earthquakes were due to slip on a fault, but they intuitively thought for many years that such slip was equivalent to a single couple (composed of two forces corresponding to the motions on opposite sides of the fault). An intuitive approach is often dangerous in elastodynamics. On the other hand, some of those who advocated the double-couple theory thought that an earthquake must be voluminal collapse under pre-existing shear stress. The fault theory of earthquake sources (now recognized as the equivalent of a double couple) has gained strong

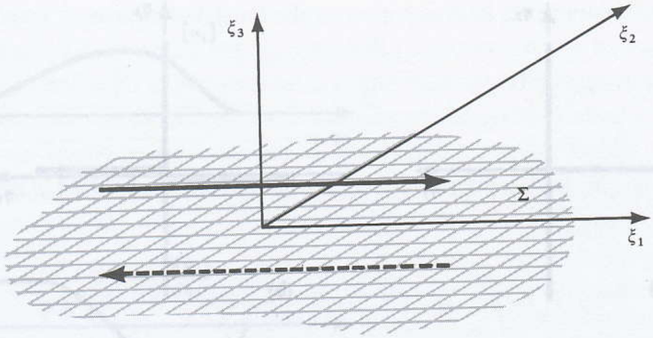

FIGURE **3.2**
A fault surface Σ within an isotropic medium is shown lying in the $\xi_3 = 0$ plane. Slip is presumed to take place in the ξ_1-direction across Σ, as shown by the heavy arrows. Motion on the side Σ^+ (i.e., $\xi_3 = 0^+$) is along the direction of ξ_1 increasing, and on the side Σ^- is along ξ_1 decreasing.

support from increasing amounts of data obtained very close to the source region, as well as support from the radiation patterns observed at great distances.

As shown in Figure 3.2, we shall take the fault Σ to lie in the plane $\xi_3 = 0$, so that $\nu_1 = \nu_2 = 0$. For the case that we are calling "fault slip", $[\mathbf{u}]$ is parallel to Σ and so $[\mathbf{u}]$ has no component in the ξ_3-direction. Let ξ_1 be the direction of slip, so that $[u_2] = [u_3] = 0$. Then the body-force equivalent, from (3.5), reduces to

$$f_p(\eta, \tau) = \iint_\Sigma [u_1(\xi, \tau)]\, c_{13pq} \frac{\partial}{\partial \eta_q} \delta(\eta - \xi)\, d\xi_1\, d\xi_2.$$

In isotropic (though still possibly inhomogeneous) media, we can find from (2.33) that all c_{13pq} vanish, except $c_{1313} = c_{1331} = \mu$. Hence

$$\begin{aligned} f_1(\eta, \tau) &= -\iint_\Sigma \mu(\xi)\, [u_1(\xi, \tau)]\, \delta(\eta_1 - \xi_1)\delta(\eta_2 - \xi_2) \frac{\partial}{\partial \eta_3} \delta(\eta_3)\, d\xi_1\, d\xi_2, \\ f_2(\eta, \tau) &= 0, \\ f_3(\eta, \tau) &= -\iint_\Sigma \mu\, [u_1]\, \frac{\partial}{\partial \eta_1} \delta(\eta_1 - \xi_1)\delta(\eta_2 - \xi_2)\delta(\eta_3)\, d\xi_1\, d\xi_2. \end{aligned} \tag{3.10}$$

First, let us look at f_1, which we shall find represents a system of single couples (forces in $\pm\eta_1$-direction, arm along η_3-direction, moment along η_2-direction) distributed over Σ. The integral above yields

$$f_1(\eta, \tau) = -\mu(\eta)\, [u_1(\eta, \tau)]\, \frac{\partial}{\partial \eta_3} \delta(\eta_3). \tag{3.11}$$

As shown in Figure 3.3, this component may be thought of as point forces distributed over the plane $\eta_3 = 0^+$ and opposed forces distributed over the plane $\eta_3 = 0^-$.

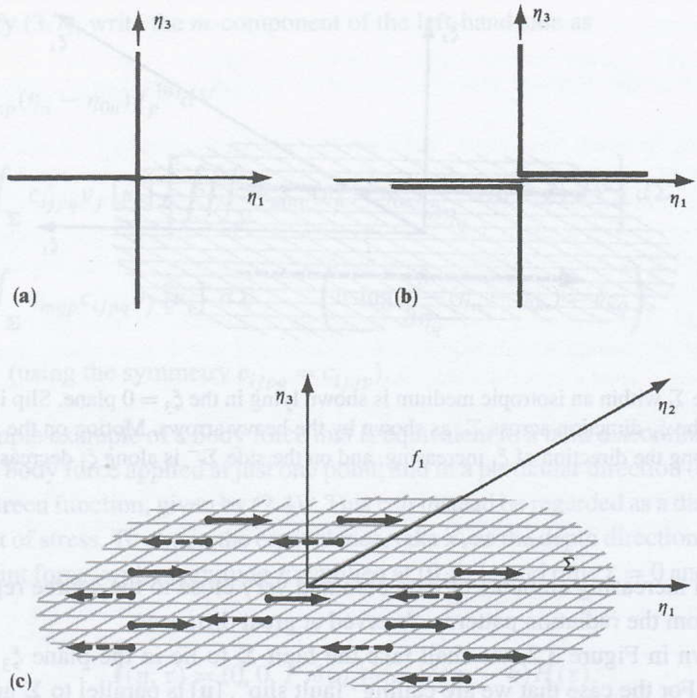

FIGURE **3.3**
Interpretive diagrams for the first component, f_1, of the body-force equivalent to fault slip of the type shown in Figure 3.2. **(a)** The spike $\left(-\delta(\eta_3), 0, 0\right)$ is plotted against η_3. (That is, a spike in the $-\eta_1$-direction, acting at $\eta_3 = 0$.) **(b)** The derivative $\left((-\partial/\partial\eta_3)\delta(\eta_3), 0, 0\right)$ is plotted against η_3. The body force $(f_1, 0, 0)$ is proportional to this quantity (see equation (3.11)). **(c)** Heavy arrows show the distribution of f_1 over the Σ^+ side of Σ and over the Σ^- side (broken arrows). This is the body-force component that would intuitively be expected in any body-force model of the motions shown in Figure 3.2.

The total force due to f_1 vanishes (see discussion of (3.6)), but the moment of this force component alone does not. The total moment about the η_2-axis is

$$\iiint_V \eta_3 f_1\, dV = -\iiint_V \eta_3 \mu \left[u_1\right] \frac{\partial}{\partial \eta_3}\delta(\eta_3)\, d\eta_1\, d\eta_2\, d\eta_3 = \iint_\Sigma \mu \left[u_1(\xi, \tau)\right]\, d\Sigma.$$

If slip is averaged over Σ to define the quantity

$$\overline{u}(\tau) = \frac{\iint_\Sigma \left[u_1(\xi, \tau)\right]\, d\Sigma}{A},$$

where $A = \iint_\Sigma d\Sigma$ is the fault area, and if the fault region is homogeneous (so that μ is constant), then the total moment about the η_2-axis due to $f_1(\xi, \tau)$ is simply $\mu\overline{u}A$ along the direction of η_2 increasing.

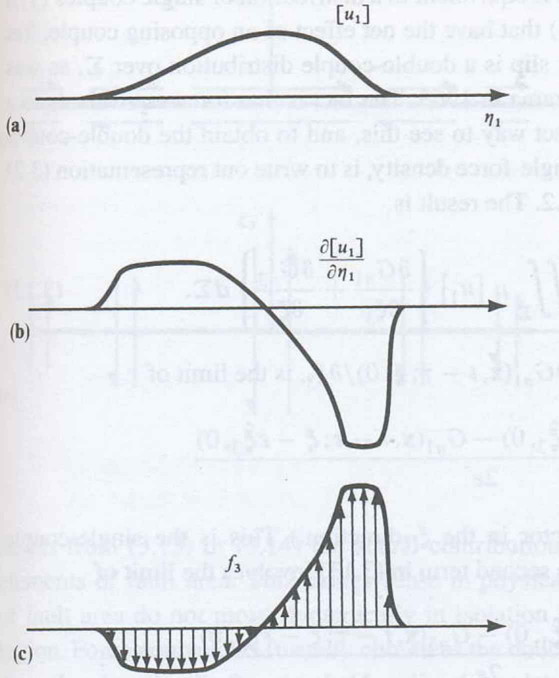

FIGURE **3.4**
Interpretive diagrams for the third component, f_3, of the body-force equivalent to fault slip $[u_1]$. **(a)** An assumed variation of slip $[u_1]$ with η_1, at fixed η_2 and τ. **(b)** The corresponding derivative $\partial [u_1] / \partial \eta_1$. **(c)** The distribution of single forces f_3 with varying η_1 (see equation (3.12)). This distribution will clearly yield a net couple, with moment in the η_2-direction.

The body-force equivalent, given in (3.10), also involves f_3, and we shall find that this represents a system of single forces. Taking the η_1-derivative outside the integration, we find

$$f_3(\eta, \tau) = -\frac{\partial}{\partial \eta_1} \{\mu [u_1(\eta, \tau)]\} \, \delta(\eta_3). \tag{3.12}$$

Although this component is not itself a couple at each point on Σ, in the sense that we have shown f_1 to be a couple, the whole distribution of f_3 across Σ does have a net moment. Figure 3.4 shows how f_3 can reverse direction at different points of Σ. The total moment about the η_2-axis is

$$\iiint_V \varepsilon_{213} \eta_1 f_3 \, dV = \iiint_V \eta_1 \frac{\partial}{\partial \eta_1} \{\mu [u_1]\} \, \delta(\eta_3) \, d\eta_1 \, d\eta_2 \, d\eta_3$$
$$= \iint_\Sigma \xi_1 \frac{\partial}{\partial \xi_1} \{\mu [u_1]\} \, d\xi_1 \, d\xi_2 = -\iint_\Sigma \mu [u_1] \, d\xi_1 \, d\xi_2.$$

(This last equality follows from an integration by parts, using a fault surface Σ defined to have $[\mathbf{u}] = \mathbf{0}$ around its perimeter.) In a homogeneous source region, it follows that the total moment due to f_3 is $-\mu \bar{u} A$, which is equal in magnitude to the total moment of f_1, but acts in the opposite direction. We obtained this result in more general form in (3.7), but have found here the two canceling contributions that arise.

We have now shown that fault slip is equivalent to a distribution of single couples (f_1), plus a distribution of single forces (f_3) that have the net effect of an opposing couple. Yet the classical force equivalent for fault slip is a double-couple distribution over Σ, as was first shown for a finite fault by Maruyama in 1963. The fact is that force equivalents to a given fault slip are not unique. A direct way to see this, and to obtain the double-couple density as well as the single-couple/single-force density, is to write out representation (3.2) for the fault slip described in Figure 3.2. The result is

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\tau \iint_{\Sigma} \mu\,[u_1] \left\{ \frac{\partial G_{n1}}{\partial \xi_3} + \frac{\partial G_{n3}}{\partial \xi_1} \right\} d\Sigma. \tag{3.13}$$

The first term here in curly brackets, $\partial G_{n1}(\mathbf{x}, t - \tau; \xi, 0)/\partial \xi_3$, is the limit of

$$\frac{G_{n1}(\mathbf{x}, t - \tau; \xi + \varepsilon\hat{\xi}_3, 0) - G_{n1}(\mathbf{x}, t - \tau; \xi - \varepsilon\hat{\xi}_3, 0)}{2\varepsilon}$$

as $\varepsilon \to 0$. (We take $\hat{\xi}_i$ as a unit vector in the ξ_i-direction.) This is the single-couple distribution shown in Figure 3.5a. The second term in (3.13) involves the limit of

$$\frac{G_{n3}(\mathbf{x}, t - \tau; \xi + \varepsilon\hat{\xi}_1, 0) - G_{n3}(\mathbf{x}, t - \tau; \xi - \varepsilon\hat{\xi}_1, 0)}{2\varepsilon},$$

and this single-couple distribution is shown in Figure 3.5b. These two systems form a double-couple distribution, and we must ask why the earlier set of body-force equivalents we derived, (3.10), made up a single couple plus a single force. The answer can be seen if one term in (3.13) is integrated by parts, giving

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\tau \iint_{\Sigma} \mu \left([u_1] \frac{\partial G_{n1}}{\partial \xi_3} - \left[\frac{\partial u_1}{\partial \xi_1} \right] G_{n3} \right) d\Sigma. \tag{3.14}$$

This force system is illustrated in Figure 3.6; clearly it is the same as the system we found first of all, shown in Figures 3.3 and 3.4. There is always a single couple (f_1, Figs. 3.3, 3.5a, and 3.6a) made up of forces in the same direction as fault-surface displacements (Fig. 3.2). But a complete equivalent to fault slip has another part, which may be regarded as a distribution of single forces (f_3, Figs. 3.4 and 3.6b), a distribution of single couples (Fig. 3.5b), or an appropriate linear combination of these alternative extremes. For a given element of area $d\Sigma$ on the fault, these force systems are physically quite different: from the integrand in representation (3.13), there appears to be no force or moment acting on $d\Sigma$; but from (3.14), there does appear to be both force and moment acting on $d\Sigma$, although we showed earlier that f_1 and f_3 integrate to give zero net force and zero net moment on the whole of Σ.

We have brought out these results in some detail, because they show the limited utility of force equivalents for studying the actual forces occurring in dynamic processes of fault slip. Whereas the body-force equivalent to fault slip *is* unique (see Box 3.3), force equivalents in the sense of this section are *not* unique (force equivalents in this case being force/unit area on a finite fault). It is the whole fault surface that is radiating seismic waves, and we cannot

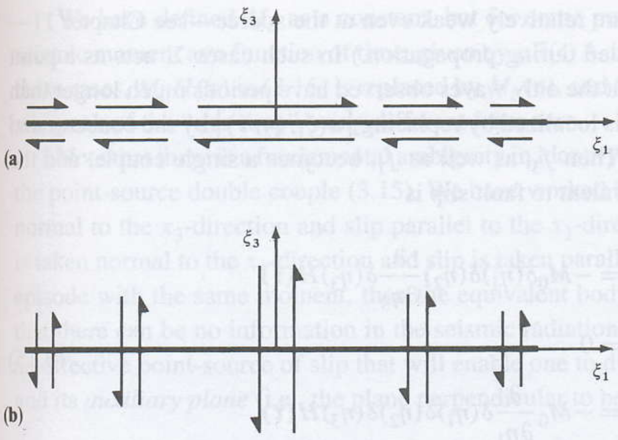

FIGURE 3.5
The radiation from these two distributions is the same as the radiation from slip on a fault. In this sense, these two single-couple distributions, taken together, are equivalent to fault slip. Note that there is no net couple, and no net force, acting on any element of area in the fault plane ($\xi_3 = 0$).

assess from (3.13) or (3.14) the actual contribution made to the radiation by individual elements of fault area. This makes sense in physical terms, because individual elements of fault area do not move dynamically in isolation from neighboring parts of the source region. Force equivalents (usually chosen as the double-couple distribution) find their main use only when the slip function $[\mathbf{u}(\xi, \tau)]$ has been determined (or guessed), and then they are important because they enable one to compute the radiation by weighting Green functions.

At great distance from a rupturing fault, it often occurs that the only waves observed are those with wavelengths much greater than linear dimensions of Σ, the causative fault.

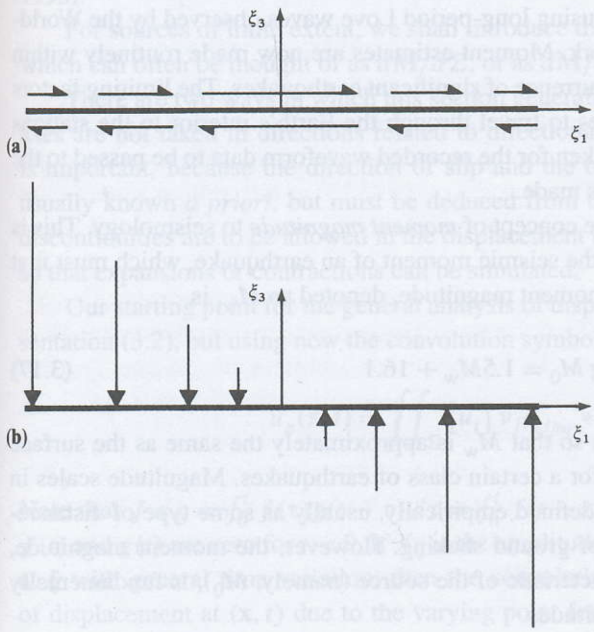

FIGURE 3.6
Another force system that is equivalent to fault slip (compare with Fig. 3.5). (a) and (b) here constitute a single-couple plus single-force system, which has zero total couple and zero total force for the whole fault surface. But individual elements of area are acted on by a couple and a force.

(Higher frequency components are relatively weak even at the source—see Chapter 11—and are more effectively attenuated during propagation.) In such cases Σ acts as a point source, and if we also assume that the only waves observed have periods much longer than the source duration then the slip is localized by replacing $[\mathbf{u}(\xi_1, \xi_2, \tau)]$ by the concentrated distribution $\bar{\mathbf{u}}A\delta(\xi_1)\delta(\xi_2)H(\tau)$. Then f_3, as well as f_1, becomes a single couple, and the double-couple point source equivalent to fault slip is

$$\begin{aligned} f_1(\eta, \tau) &= -M_0\delta(\eta_1)\delta(\eta_2)\frac{\partial}{\partial \eta_3}\delta(\eta_3)H(\tau) \\ f_2(\eta, \tau) &= 0 \\ f_3(\eta, \tau) &= -M_0\frac{\partial}{\partial \eta_1}\delta(\eta_1)\delta(\eta_2)\delta(\eta_3)H(\tau), \end{aligned} \tag{3.15}$$

where

$$M_0 = \mu\bar{u}A = \mu \times \text{average slip} \times \text{fault area}. \tag{3.16}$$

We call M_0 the *seismic moment*. It is perhaps the most fundamental parameter we can use to measure the strength of an earthquake caused by fault slip. Measured values of M_0 range from about 10^{30} dyn-cm (1960 Chilean earthquake, 1964 Alaskan earthquake) down to around 10^{12} dyn-cm for microearthquakes, and 10^5 dyn-cm for microfractures in laboratory experiments on loaded rock samples. Even for geophysics, twenty-five orders of magnitude is an exceptionally large range to be spanned by a single physical variable. The first person to obtain the double-couple equivalence for an effective point source of slip was Vvedenskaya (1956). The first estimate of seismic moment was made by Aki (1966) for the Niigata earthquake of 1964, using long-period Love waves observed by the World-Wide Standard Seismograph Network. Moment estimates are now made routinely within hours, or even minutes, after the occurrence of significant earthquakes. The limiting factors are the time taken for seismic waves to travel through the Earth's interior to the stations whose data are used; and the time taken for the recorded waveform data to be passed to the processing site where the estimate is made.

Kanamori (1977) introduced the concept of *moment magnitude* to seismology. This is simply a magnitude scale based on the seismic moment of an earthquake, which must first be estimated. His definition of the moment magnitude, denoted as M_w, is

$$\log M_0 = 1.5M_\text{w} + 16.1 \tag{3.17}$$

in which the constants were chosen so that M_w is approximately the same as the surface wave magnitude (see Appendix 2) for a certain class of earthquakes. Magnitude scales in seismology have traditionally been defined empirically, usually as some type of distance-corrected measure of the strength of ground shaking. However, the moment magnitude, being derived from a physical characteristic of the source (namely, M_0), is fundamentally different from these empirical magnitudes.

We have defined M_0 as a constant, but for some purposes it is useful to evaluate the seismic moment as a function of time, given by $\mu\overline{u}(t)A$, in which $\overline{u}$ is averaged at time t. In these cases, $M_0 H(\tau)$ in (3.15) is replaced by $M_0(\tau)$, and (in the terminology of Chapter 10) we speak of the "rise time" being different from zero.

Note that there is a fundamental ambiguity in identifying the fault plane associated with the point-source double couple (3.15). We have worked in this section with a fault surface normal to the x_3-direction and slip parallel to the x_1-direction. If the fault surface instead is taken normal to the x_1-direction and slip is taken parallel to the x_3-direction in a faulting episode with the same moment, then the equivalent body force is again (3.15). It follows that there can be no information in the seismic radiation or static displacement field from an effective point-source of slip that will enable one to distinguish between the fault plane and its *auxiliary plane* (i.e., the plane perpendicular to both the fault and the slip).

3.3 General Analysis of Displacement Discontinuities across an Internal Surface Σ

In this section we introduce the seismic *moment tensor*, **M**. This is a quantity that depends on source strength and fault orientation, and it characterizes all the information about the source that can be learned from observing waves whose wavelengths are much longer than the linear dimensions of Σ. In this case, the source is effectively a point source with an associated radiation pattern, and the moment tensor can often be estimated in practice for a given earthquake by using long-period teleseismic data. In practice, seismologists usually use moment tensors that are confined to sources having a body-force equivalent given by pairs of forces alone (couples, vector dipoles). Such sources include geologic faults (shearing) and explosions (expansion), with **M** as a second-order tensor. For forces differentiated more than once, sources can be characterized by higher order moment tensors (see Julian *et al.*, 1998).

For sources of finite extent, we shall introduce the seismic *moment density tensor*, **m**, which can often be thought of as $d\mathbf{M}/d\Sigma$, or as $d\mathbf{M}/dV$ for a volume source.

There are two ways in which this section generalizes Section 3.2. First, the coordinate axes are not taken in directions related to directionalities of the source. (This generality is important, because the direction of slip and the orientation of the fault plane are not usually known *a priori*, but must be deduced from the radiated seismic waves.) Second, discontinuities are to be allowed in the displacement component normal to the fault plane, so that expansions or contractions can be simulated.

Our starting point for the general analysis of displacement discontinuities is the representation (3.2), but using now the convolution symbol $*$ so that

$$u_n(\mathbf{x}, t) = \iint_\Sigma [u_i]\, \nu_j c_{ijpq} * \frac{\partial}{\partial \xi_q} G_{np}\, d\Sigma. \tag{3.18}$$

Note that $f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau$ if $f(t)$ and $g(t)$ are zero for $t < 0$. If X_0 is the amplitude of a force applied in the p-direction at ξ with general time variation, then the convolution $X_0 * G_{np}$ gives the n-component of displacement at $(\mathbf{x}, t)$ due to the varying point force at ξ. More generally, if the force

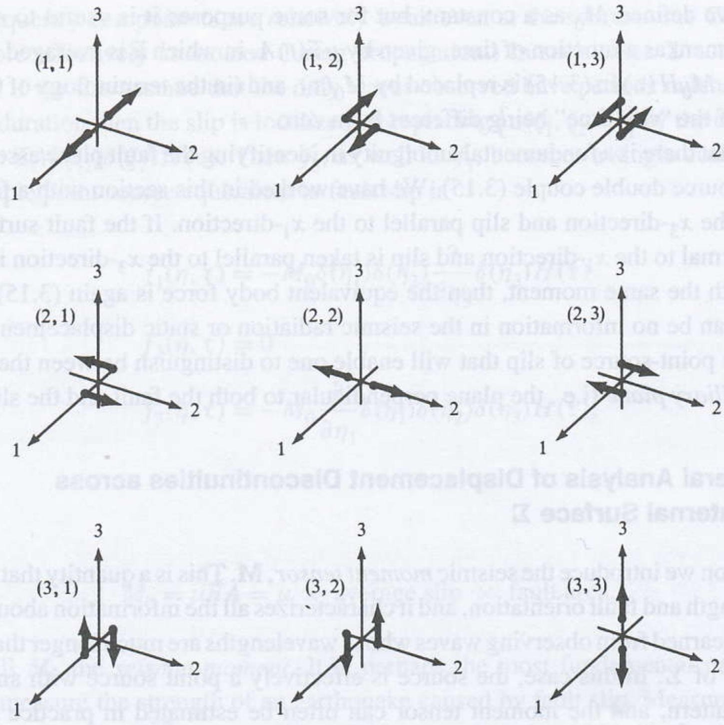

FIGURE **3.7**
The nine possible couples that are required to obtain equivalent forces for a generally oriented displacement discontinuity in anisotropic media.

applied at ξ is $\mathbf{F}(\xi, \tau)$, then we can sum over p and write $F_p * G_{np}$ for the n-component of displacement at $(\mathbf{x}, t)$. For displacement discontinuities as in (3.18), there are instead derivatives of G_{np} with respect to the source coordinates ξ_q. Such a derivative, we saw in Section 3.2, can be thought of physically as the equivalent of having a single couple (with arm in the ξ_q-direction) on Σ at ξ. The sum over q in (3.18) is then telling us that each displacement component at $\mathbf{x}$ is equivalent to the effect of a sum of couples distributed over Σ.

For three components of force and three possible arm directions, there are nine generalized couples, as shown in Figure 3.7. Thus the equivalent surface force corresponding to an infinitesimal surface element $d\Sigma(\xi)$ can be represented as a combination of nine couples. In general, we need "couples" with force and arm in the same direction (cases (1, 1), (2, 2), (3, 3) of Fig. 3.7), and these are sometimes called *vector dipoles*.

Since $[u_i]\,\nu_j c_{ijpq} * \partial G_{np}/\partial \xi_q$ in (3.18) is the n-component of the field at $\mathbf{x}$ due to couples at ξ, it follows that $[u_i]\,\nu_j c_{ijpq}$ is the strength of the (p, q) couple. The dimensions of $[u_i]\,\nu_j c_{ijpq}$ are moment per unit area, and this makes sense because the contribution from ξ has to be a surface density, weighted by the infinitesimal area element $d\Sigma$ to give a moment contribution. We define

$$m_{pq} = [u_i]\,\nu_j c_{ijpq} \tag{3.19}$$

to be the components of the *moment density tensor*, $\mathbf{m}$. In terms of this symmetric tensor, which is time dependent, the representation theorem for displacement at $\mathbf{x}$ due to general displacement discontinuity $[\mathbf{u}(\xi,\tau)]$ across Σ is

$$u_n(\mathbf{x},t) = \iint_\Sigma m_{pq} * G_{np,q}\, d\Sigma. \tag{3.20}$$

When we have learned more about the Green function (in Chapter 4), we shall find that the time dependence of the integrand in (3.20) is quite simple, because if $\mathbf{x}$ is many wavelengths away from ξ, then convolution with $\mathbf{G}$ gives a field at $(\mathbf{x}, t)$ that depends on what occurs at ξ only at "retarded time," i.e., t minus some propagation time between ξ and $\mathbf{x}$.

For an isotropic body, it follows from (2.33) and (3.19) that

$$m_{pq} = \lambda\nu_k\,[u_k(\xi,\tau)]\,\delta_{pq} + \mu\left(\nu_p\left[u_q(\xi,\tau)\right] + \nu_q\left[u_p(\xi,\tau)\right]\right). \tag{3.21}$$

Further, if the displacement discontinuity (or slip) is parallel to Σ at ξ, the scalar product $\nu \cdot [\mathbf{u}]$ is zero and

$$m_{pq} = \mu\left(\nu_p\left[u_q\right] + \nu_q\left[u_p\right]\right). \tag{3.22}$$

In the case of Σ lying in the plane $\xi_3 = 0$, with slip only in the ξ_1-direction, we have the source model considered in Section 3.2, and for this the moment density tensor is

$$\mathbf{m} = \begin{pmatrix} 0 & 0 & \mu\,[u_1(\xi,\tau)] \\ 0 & 0 & 0 \\ \mu\,[u_1(\xi,\tau)] & 0 & 0 \end{pmatrix},$$

which is the now familiar double couple.

In the case of a tension crack in the $\xi_3 = 0$ plane, only the slip component $[u_3]$ is nonzero, and from (3.21) we find

$$\mathbf{m} = \begin{pmatrix} \lambda\,[u_3(\xi,\tau)] & 0 & 0 \\ 0 & \lambda\,[u_3(\xi,\tau)] & 0 \\ 0 & 0 & (\lambda+2\mu)\,[u_3(\xi,\tau)] \end{pmatrix}.$$

Thus a tension crack is equivalent to a superposition of three vector dipoles with magnitudes in the ratio $1 : 1 : (\lambda + 2\mu)/\lambda$ (see Fig. 3.8).

The above results have been developed for a fault plane Σ of finite extent, but in practice the seismologist often has data that are good only at periods for which the whole of Σ is effectively a point source. For these waves, the contributions from different surface elements $d\Sigma$ are all approximately in phase, and the whole surface Σ can be considered as a system of couples located at a point, say the center of Σ, with *moment tensor* equal to the integral of moment density over Σ. Thus, for an effective point source,

$$u_n(\mathbf{x},t) = M_{pq} * G_{np,q}, \tag{3.23}$$

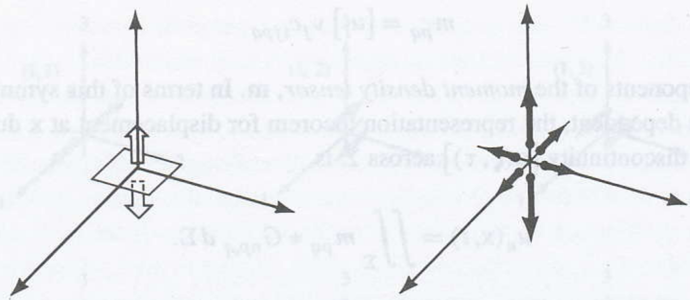

FIGURE **3.8**
The body-force equivalent for a tension crack in an isotropic medium.

where the moment tensor components are

$$M_{pq} = \iint_{\Sigma} m_{pq}\, d\Sigma = \iint_{\Sigma} [u_i]\, \nu_j c_{ijpq}\, d\Sigma, \qquad \text{i.e.,} \qquad m_{pq} = \frac{dM_{pq}}{d\Sigma}. \tag{3.24}$$

In (3.23) we have one of the most important equations of this chapter. Later in this book, we shall evaluate the Green function and the different waves it contains. Thus in Chapter 4 we shall use ray theory for **G** and interpret (3.23) in terms of body waves excited by given **M** (equation (4.96)). In Chapter 7, we shall find the surface waves excited by **M** (equations (7.148)–(7.151)), and in Chapter 8 the normal modes of the whole Earth (8.38).

In terms of seismic moment M_0, and with the choice of coordinate axes made in Section 3.2, the moment tensor for an effective point source of slip is

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & M_0 \\ 0 & 0 & 0 \\ M_0 & 0 & 0 \end{pmatrix}. \tag{3.25}$$

Equations (3.24) justify the name "moment tensor density" for **m**. In the case of a finite source, we can now interpret the representation (3.20) as an areal distribution of point sources, each point having the moment tensor $\mathbf{m}\, d\Sigma$.

We conclude this section with an interesting use of "seismic moment," suggested by Brune (1968), involving the kinematic motions of tectonic plates. Such motions lead frequently to a type of regional seismicity in which many different earthquakes share the same fault plane (although any one event will involve slip over only a part of the whole fault area). If M_0^i is the seismic moment of the ith earthquake in a series of N earthquakes in time interval ΔT, it follows from the definition of M_0^i that the total slip due to the whole series is

$$\Delta U = \frac{\sum_{i=1}^{N} M_0^i}{\mu S}, \tag{3.26}$$

BOX **3.2**

On uses of the word "moment" in seismic source theory

In rotational mechanics, it is often enough to speak of a couple possessing the qualities of magnitude and a single direction. The magnitude of a couple is then a scalar quantity called the moment. In our study of displacement discontinuities, however, and body-force equivalents, we imply more directional qualities behind the word "couple" than is the case in rigid-body rotational mechanics. For us, "couple" involves the directions of both force and lever arm. A result of this is that the quantity "moment" jumps up from scalar to tensor.

Second-order Cartesian tensors in mathematical physics are usually quantities that relate one physical vector to another. One example is given by equation (2.16), in which the stress tensor is a device for obtaining traction from the vector orientation of an area element. Another example is the inertia tensor $\mathbf{I}$, which gives angular momentum $\mathbf{h}$ from angular velocity ω via $h_i = I_{ij}\omega_j$. In seismic source theory, however, the moment tensor is an input rather than a filter, and it is operated on by a third-order tensor to yield vector displacement (see (3.20) and (3.23)).

where S is the total area broken in the series. ΔU is averaged over all of S, and all the terms in the right-hand side of (3.26) can be estimated. If all the plate motion occurs seismically, and if the seismicity during ΔT is representative of the activity on that plate margin for longer time scales, then $\Delta U / \Delta T$ is an estimate of the relative velocity of the plates, regarded as slow-moving rigid bodies, and it can be obtained from seismic data alone.

3.4 Volume Sources: Outline of the Theory and Some Simple Examples

In order to develop equations for seismic waves from buried explosions or from rapid phase transformations, it is necessary to introduce the concept of a volume source. We shall describe such a source in terms of a transformational (or stress-free) strain introduced in the source volume, and shall develop properties of an associated seismic moment tensor.

Let us illustrate this concept by a set of imaginary cutting, straining, and welding operations described by Eshelby (1957). First, we separate the source material by cutting along a closed surface Σ that surrounds the source, and we remove the source volume (the "inclusion") from its surroundings (the "matrix"). We suppose that the material removed is held in its original shape by tractions having the same value over Σ as the tractions imposed across Σ by the matrix before the cutting operation. Second, we let the source material undergo transformational strain Δe_{rs}. By this, we mean that Δe_{rs} occurs without changing the stress within the inclusion, hence the name "stress-free strain." It is this strain that characterizes the seismic source. Processes that can be described by stress-free strain include phase transformation, thermal expansion, and some plastic deformations. Stress-free strain is a static concept. Third, we apply extra surface tractions that will restore the source volume to its original shape: this will result in an additional stress field $-c_{pqrs}\,\Delta e_{rs} = -\Delta\tau_{pq}$ throughout the inclusion, and the additional tractions applied on its surface Σ are $-c_{pqrs}\,\Delta e_{rs}\nu_q$, where ν_q is the outward normal on Σ. Since $\Delta\tau_{pq}$ is a static field, $\Delta\tau_{pq,q} = 0$. The stress in the matrix is still unchanged, being held at its original value

BOX **3.3**

Body-force equivalents and the seismic moment tensor

For a general displacement discontinuity across Σ, it follows from (3.5) that

$$f_p = -\frac{\partial}{\partial \eta_q}\left\{[u_i]\nu_j c_{ijpq}\delta(\Sigma)\right\},$$

where, by $\delta(\Sigma)$, we mean a one-dimensional spatial Dirac delta function that is zero off Σ. Thus, if Σ lies in the plane $\eta_3 = 0$, $\delta(\Sigma) = \delta(\eta_3)$ for points (η_1, η_2) on Σ.

It must be emphasized that $\mathbf{f}$ is a force *per unit volume*, and it is unique. (Once $[u_i]$ is given on Σ, then $\mathbf{u}$ is determined everywhere, and $\mathbf{f} = \mathbf{L}(\mathbf{u})$, where $\mathbf{L}$ is given in Box 2.4.) The ambiguities mentioned in Section 3.2 arise only when equivalent *surface* forces are sought. Thus the above formula for f_p does not give a distribution of couples and dipoles. Such a distribution arises only after the displacement representation $\iiint_V G_{np}\{f_p\}\,dV$ has been integrated by parts and the η_3 integration completed to give (3.18), which may then be interpreted in terms of equivalent surface forces. These are nonunique—see (3.13) and (3.14)—but a surface distribution of couples and vector dipoles is always possible.

We have introduced the seismic moment tensor in the form $M_{pq} = \iint_\Sigma [u_i]\,\nu_j c_{ijpq}\,d\Sigma$, but from the above formula for body force it is easy to show that

$$M_{pq} = \iiint_V f_p \eta_q \, dV(\eta).$$

This result can be used to extend the definition of $\mathbf{M}$, since it can be used for any body-force distribution, and not just for the body-force equivalent to a displacement discontinuity. With this definition, the moment (in the ordinary sense of rotational mechanics) of body-forces $\mathbf{f}$ about the ith axis is $\iiint_V \varepsilon_{ijk}\eta_j f_k\,dV = \varepsilon_{ijk}M_{kj}$, which is zero whenever the moment tensor is symmetric (e.g., in (3.24)).

by tractions imposed across the internal surface Σ, and having the same value as tractions imposed on the matrix by the inclusion before it was cut out. Fourth, we put the inclusion back in its hole (which is exactly the correct shape) and weld the material across the cut. The traction on Σ^- is now an amount $-c_{pqrs}\,\Delta e_{rs}\nu_q$ greater than that on Σ^+, leading to a traction discontinuity (in the $\boldsymbol{\nu}$-direction) of amount $+c_{pqrs}\,\Delta e_{rs}\nu_q$. This traction is due to applied surface forces that are external to the source and which act on the inclusion to maintain its correct shape. Fifth, we release the applied surface forces over Σ^-. Since traction is actually continuous across Σ, this amounts to imposing an apparent traction discontinuity of $-(c_{pqrs}\,\Delta e_{rs})\nu_q$. The elastic field produced in the matrix by the whole process is that due to the apparent traction discontinuity across Σ.

The above procedure can be extended to a dynamic case of seismic wave generation, since, at any given time, a transformational strain Δe_{rs} can be defined for the unrestrained material. For each instant, it is still true that $\Delta\tau_{pq,q} = 0$ because stress-free strain (and the stress derived from it) is a static concept. The seismic displacement generated by the

BOX **3.4**

The strain energy released by earthquake faulting

Within a medium that initially has a static stress field $\boldsymbol{\sigma}^0$, we suppose that a displacement discontinuity develops across an internal surface Σ. This leads to a displacement field $\mathbf{u}(\mathbf{x}, t)$, measured with reference to the initial configuration, and from $\mathbf{u}$ we can determine the additional time-dependent strain and the additional stress $\boldsymbol{\tau}$. Then the total stress is $\boldsymbol{\sigma} = \boldsymbol{\sigma}^0 + \boldsymbol{\tau}$, and after all motions have died down the new static stress field is $\boldsymbol{\sigma}^1$. If ΔE is defined as the change in strain energy throughout the medium, from its initial static configuration to its final static configuration, it can be shown that

$$\Delta E = -\tfrac{1}{2} \int_\Sigma [u_i] \, (\sigma_{ij}^0 + \sigma_{ij}^1)\nu_j \, d\Sigma, \tag{1}$$

where $[\mathbf{u}]$ is the final offset. (See Fig. 3.1 for definitions of [ ] and ν.) Equation (1) is known as the Volterra relation (Steketee, 1958; Savage, 1969a).

This result (which we derive below) can be simply restated in terms of work apparently done by tractions on the fault surface. We can say from (1) that the drop in strain energy throughout the medium, $-\Delta E$, is the positive quantity obtained by imagining a quasi-static growth of traction that is linear with offset:

$$T = T^0 + (T^1 - T^0)\frac{U}{[u]} \qquad \text{for} \qquad 0 \le U \le [u] \tag{2}$$

(for each component of traction T and displacement U). Integrating from 0 to $[u]$ to get the total work done on Σ then gives (1).

Several points now need to be made about this relation between ΔE and the average stress.

The liberated energy, $-\Delta E$, supplies the work actually done on the two faces Σ^+ and Σ^- as they grind past each other, plus the work done in initiating the process of fracture. We discuss these two types of work in Chapter 11. Moreover, $-\Delta E$ supplies the seismic energy E_s that is radiated away from the source region. It is natural to introduce the *seismic efficiency*, η, as the ratio $E_s/(-\Delta E)$. Then

$$E_s = -\eta \, \Delta E = \tfrac{1}{2}\eta \int_\Sigma [u_i] \, (\sigma_{ij}^0 + \sigma_{ij}^1)\nu_j \, d\Sigma. \tag{3}$$

If the average of the two static tractions does not vary strongly over Σ, then for the type of tangential slip shown in Figure 3.2 we see that (3) can be expressed in terms of the moment $M_0 = \mu \int_\Sigma [u_1] \, d\Sigma$. This gives

$$E_s = \eta M_0 \overline{\sigma}/\mu, \tag{4}$$

where $\overline{\sigma} = \frac{1}{2}(\sigma_{13}^0 + \sigma_{13}^1)$.

From estimates that can be made of E_s, M_0, and μ, it thus becomes possible from (4) to estimate the product $\eta\overline{\sigma}$, called the *apparent stress* by Wyss and Brune (1968, 1971). The reason for this name is that $\eta\overline{\sigma}$ would be the stress that *appears* to be acting on the fault, if we make the assumption that the observed radiated energy is equal to the liberated strain energy. (The assumption here is not a good one. The seismic efficiency is at most a few percent, so that only a small fraction of the liberated energy is radiated as seismic waves.)

(continued)

BOX **3.4** *(continued)*

Since the slip function $[\mathbf{u}(\xi, t)]$ in (3.18) determines all displacements (and hence strain and stress increments) throughout the medium, it also determines the stress drop, $\sigma^0 - \sigma^1$. But there is no way one can work purely from observations of the radiated field $\mathbf{u}(\mathbf{x}, t)$ and learn anything about the absolute level of stress in the source region. Putting this another way, and using (1), one can make the following statement. If the same slip function $[\mathbf{u}(\xi, t)]$ occurs on Σ in two different faulting events with different initial stresses, then all the seismic displacements will be the same for the two events: but the strain energies liberated for the two events may be quite different.

It remains, then, to prove our main result (1). This is a formula of great generality, and a correct derivation can be given by considering the quasi-static deformation we described in (2). We shall give an explicit proof for the special case in which the internal strain energy $\mathcal{U}$ is given by a strain–energy function $\mathcal{W}$ (see Section 2.2). Further, we assume there is an accessible reference state of zero stress and zero strain. The initial stresses and strains just prior to faulting are σ^0_{ij} and e^0_{ij}, and $\mathbf{u}$ is measured from this state.

From (2.32) applied to the total stresses and strains, we get

$$\begin{aligned}\mathcal{W} &= \tfrac{1}{2}(\sigma^0_{ij} + \tau_{ij})(e^0_{ij} + u_{i,j}) \qquad \text{(using symmetry of } \sigma_{ij}\text{)}\\ &= \mathcal{W}^0 + \tfrac{1}{2}\sigma_{ij}u_{i,j} + \tfrac{1}{2}c_{ijkl}u_{k,l}e^0_{ij}\\ &= \mathcal{W}^0 + \tfrac{1}{2}\sigma_{ij}u_{i,j} + \tfrac{1}{2}\sigma^0_{kl}u_{k,l} \qquad \text{(using (2.30)).}\end{aligned}$$

Thus the increase in internal energy in the new static configuration is

$$\Delta E = \int_V (\mathcal{W}^1 - \mathcal{W}^0)\, dV = \tfrac{1}{2}\int_V (\sigma_{ij} + \sigma^0_{ij})u_{i,j}\, dV, \tag{5}$$

where V is the whole elastic volume containing Σ (see Fig. 3.1). Since σ^0_{ij} and σ^1_{ij} are static stress fields, (2.17) implies $\sigma^0_{ij,j} = \sigma^1_{ij,j} = 0$ (we assume there are no body forces). From (5), we obtain

$$\Delta E = \tfrac{1}{2}\int_V \{(\sigma^0_{ij} + \sigma^1_{ij})u_i\}_{,j}\, dV,$$

to which we can apply Gauss's divergence theorem, regarding V as the interior of $S + \Sigma^+ + \Sigma^-$. This does give (1) if S is a rigid surface, or if, like the surface of the Earth, it is free.

discontinuity in traction was given by (3.3). Putting $[T_p] = -(c_{pqrs}\ \Delta e_{rs})\nu_q$ in (3.3), we get

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\tau \iint_\Sigma c_{pqrs}\ \Delta e_{rs}\nu_q G_{np}(\mathbf{x}, t - \tau; \xi, 0)\, d\Sigma(\xi). \tag{3.27}$$

If the integrand and its derivatives with respect to ξ are continuous, we can apply the Gauss theorem to obtain

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\tau \iiint_V \frac{\partial}{\partial \xi_q}\left\{c_{pqrs}\ \Delta e_{rs} G_{np}(\mathbf{x}, t - \tau; \xi, 0)\right\} dV(\xi) \tag{3.28}$$

(V here refers only to the volume of the inclusion, i.e., the source volume). Using $\partial(c_{pqrs}\,\Delta e_{rs})/\partial\xi_q = \Delta\tau_{pq,q} = 0$, we can rewrite (3.28) and obtain

$$u_n(\mathbf{x}, t) = \iiint_V c_{pqrs}\,\Delta e_{rs} * \frac{\partial G_{np}}{\partial \xi_q}\,dV. \tag{3.29}$$

Comparing this volume integral with the surface integral in (3.18), one sees that it is natural to introduce a moment-density tensor

$$\frac{dM_{pq}}{dV} = c_{pqrs}\,\Delta e_{rs} \tag{3.30}$$

with the dimensions of moment per unit volume (compare also with (3.24)). Then

$$u_n(\mathbf{x}, t) = \iiint_V \frac{dM_{pq}}{dV} * \frac{\partial G_{np}}{\partial \xi_q}\,dV. \tag{3.31}$$

Note that $\Delta\tau_{pq} = dM_{pq}/dV$ is not the stress drop (the difference between the initial equilibrium stress and the final equilibrium stress in the source region), as is clear from its definition. The stress drop is not limited to the source volume, but $\Delta\tau_{pq}$ vanishes outside the source volume. $\Delta\tau_{pq}$ is called the "stress glut" by Backus and Mulcahy (1976).

For long waves, for which the whole of V is effectively a point source, the whole volume V can be considered a system of couples located at a point, say the center of V, with moment tensor equal to the integral of moment density over V. Thus, for an effective point source, (3.23) applies, with the moment tensor components

$$M_{pq} = \iiint_V c_{pqrs}\,\Delta e_{rs}\,dV. \tag{3.32}$$

For example, if a shear collapse occurs in a homogeneous isotropic body of volume V with the nonzero transformational strain components $\Delta e_{13} = \Delta e_{31}$, say, the moment tensor is

$$\mathbf{M} = 2\mu V \begin{pmatrix} 0 & 0 & \Delta e_{13} \\ 0 & 0 & 0 \\ \Delta e_{13} & 0 & 0 \end{pmatrix}. \tag{3.33}$$

The seismic radiation is identical to the point source equivalent to a fault slip, except that the seismic moment M_0 is given by $2\mu\,\Delta e_{13} V$. For a group of earthquakes in an intraplate seismic zone, a cumulative strain may be more meaningful than a cumulative slip given by (3.26). Kostrov (1974) suggested summing moments for a group of earthquakes sharing the same source mechanism in a given volume to find the total strain in the volume. From (3.33), the total strain ΔE_{13} may be estimated as

$$\Delta E_{13} = \frac{\sum_{i=1}^{N} M_0^i}{2\mu V}, \tag{3.34}$$

where M_0^i is the moment of the ith earthquake.

Finally, let us consider a spherical volume with radius a undergoing a transformational expansion. The stress-free strain components in this case are $\Delta e_{12} = \Delta e_{13} = \Delta e_{23} = 0$ and $\Delta e_{11} = \Delta e_{22} = \Delta e_{33} = \frac{1}{3}\ \Delta V/V$, where $\Delta V/V$ is the fractional change in volume and $V = \frac{4}{3}\pi a^3$. For this expansion in an isotropic medium, $c_{pqrs}\ \Delta e_{rs} = (\lambda + \frac{2}{3}\mu)\delta_{pq}\ \Delta V/V$ and from (3.32) we have

$$\mathbf{M} = \begin{pmatrix} (\lambda + \frac{2}{3}\mu)\Delta V & 0 & 0 \\ 0 & (\lambda + \frac{2}{3}\mu)\Delta V & 0 \\ 0 & 0 & (\lambda + \frac{2}{3}\mu)\Delta V \end{pmatrix}. \tag{3.35}$$

Thus a spherical source with transformational volume expansion is equivalent to three mutually perpendicular dipoles, as shown in Figure 3.7. In the above equation, ΔV is the stress-free volume change and should not be confused with the volume change δV of a confined source region, as discussed in Problem 3.8.

Suggestions for Further Reading

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Problems

3.1 Equations (3.26) and (3.34) are written as scalar equations, because in their derivation it has been assumed that earthquakes in a given region (on S, or within V) all have moment tensors with the same orientations.

Generalize (3.26) to a vector equation and (3.34) to a tensor equation in cases where earthquakes in the series (on S or in V) have moment tensors of arbitrary orientation. (For (3.26), however, continue to assume that the displacement discontinuity for each event is a shear and that S is planar.)

3.2 In our derivation of (3.2), we have assumed that the elastic moduli are continuous across Σ and that G_{np} and $\partial G_{np}/\partial \xi_q$ are continuous. If the elastic moduli are *not* continuous across Σ, interpret part of the integrand in (3.2) as a traction, and show that this representation is still valid, although $\partial G_{np}/\partial \xi_q$ may not be continuous across the surface. (*Note:* For purposes of defining $\mathbf{G}$, assume Σ^+ and Σ^- have been glued together. These surfaces—which can still move—then do not have relative motion.)

3.3 In the discussion following equations (3.15) and (3.16), we introduced the time-dependent seismic moment given by $M_0(t) = \mu\bar{u}(t)A$. Is $\bar{u}(t)$ here averaged over the area $A(t)$ that has ruptured at time t, or is it averaged over $A(\infty)$, the area that ultimately is ruptured during the seismic event under consideration? (*Hint:* Does it matter?)

3.4 Show that the moment tensor $\mathbf{M}$ described in terms of a double couple in Section 3.2 and equation (3.25), i.e.,

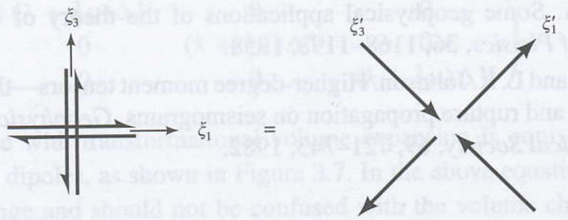
$$\mathbf{M} = \begin{pmatrix} 0 & 0 & M_0 \\ 0 & 0 & 0 \\ M_0 & 0 & 0 \end{pmatrix},$$

can equivalently be described by

$$\mathbf{M} = \begin{pmatrix} M_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -M_0 \end{pmatrix}$$

where components of $\mathbf{M}$ are now referred to the principal axes of $\mathbf{M}$ as coordinate axes. (By definition, the principal axes of a symmetric tensor are such that the off-diagonal components of the tensor, referred to these axes, are all zero.)

In terms of body-force equivalents, this result is illustrated by the following diagram:



This shows that a double couple is equivalent to a pair of vector dipoles, equal in magnitude but opposite in sign.

3.5 Show that a seismic point source described by a symmetric second-order moment tensor $\mathbf{M}$ can be thought of as an isotropic point source $\mathbf{M}_\text{I}$ plus two double couples. Is this a unique decomposition of such a point source?

Show that $\mathbf{M}$ can also be written in the form

$$\mathbf{M} = \mathbf{M}_\text{I} + (M_1 - M_2)\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + M^{\text{CLVD}}\begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

in which M_i ($i = 1, 2, 3$) are the principal moments. The last term here is called a "compensated linear vector dipole," having axial symmetry and no volume change. If $M_1 - M_2$ is the largest difference between principal moments, then M^{CLVD} quantifies the extent to which the deviatoric part of the moment tensor differs from a pure double couple.

3.6 Show that the body-force equivalent to a point source at ξ with moment tensor M_{pq} is given by

$$f_p(\mathbf{x}, t) = -M_{pq}(t)\frac{\partial}{\partial x_q}\delta(\mathbf{x} - \xi).$$

3.7 Consider a spherical cavity with radius a inside a homogeneous isotropic body. When a uniform step in pressure, $\delta p H(t)$, is applied at the surface of the cavity, spherically symmetric waves will be generated, which have displacement only in the radial direction. After the waves have passed, displacement everywhere tends to its final static value, which characterizes the final outward expansion due to the applied pressure in the cavity.

a) Use the vector wave equation of Problem 2.1 to show that this static displacement satisfies $\nabla(\nabla \cdot \mathbf{u}) = \mathbf{0}$.

b) Hence, in this problem with spherical symmetry, show that the radial displacement is proportional to $1/r^2$ (for $a \le r$, so that this is a so-called external solution).

c) Show from equations (2.50) and (2.46) that for this problem the radial stress is given by

$$\tau_{rr} = (\lambda + 2\mu)\frac{\partial u_r}{\partial r} + \frac{2\lambda}{r}u_r.$$

d) The walls of the cavity will oscillate at first, after the constant step in pressure is applied, but will eventually be displaced outward a constant amount. Let this final static displacement be δa. Show that

$$\delta p = 4\mu\frac{\delta a}{a}.$$

3.8 Suppose that a spherical volume with radius a, inside a homogeneous isotropic unbounded medium, undergoes expansion with stress-free volumetric strain given by $\Delta V/V$ where $V = \frac{4}{3}\pi a^3$. The moment tensor is given by equation (3.35), but now we shall consider the effects of the rest of the medium, which prevents the actual strain from attaining its stress-free value.

The confinement of the source region means that instead of radius a expanding to $a + \Delta a$ (where ΔV is given to first order by $4\pi a^2 \Delta a$), and being subjected to zero pressure, the final static radius is given by $a + \delta a$, subjected to pressure δp. We can build upon the results of Problem 3.7 to obtain relationships between the stress-free changes characterized by Δa and ΔV (see equation (3.35)) and the actual final static changes, δa and δV.

a) Use the method of Problem 3.7 to show that within the source region the final static radial displacement is proportional to r. (This is the so-called internal solution.) If A is the constant of proportionality such that the static radial displacement is Ar, show that the associated radial stress τ_{rr} is a constant and the final static value of pressure throughout the source region is

$$\delta p = -(3\lambda + 2\mu)A.$$

b) For this problem we can evaluate key steps in the series of cutting and welding operations first described by Eshelby and covered in Section 3.4. The static displacement of the surface of the source region, due to the effects of confinement, is from Δa to δa as pressure changes from the stress-free value (which is zero, by definition) to the final actual static value, δp. Show then that

$$\delta p = \frac{3\lambda + 2\mu}{a}(\Delta a - \delta a)$$

and hence, from a relationship given in Problem 3.7, that

$$\Delta a = \frac{\lambda + 2\mu}{\lambda + \frac{2}{3}\mu}\delta a.$$

c) As indicated in equation (3.35), the moment tensor is isotropic. In general $M_{pq} = M_0(t)\delta_{pq}$ is a function of time that depends on details of the process by which the source region undergoes its change in properties. The final static value from (3.35) is given by

$$M_0(\infty) = (\lambda + \tfrac{2}{3}\mu)\ \Delta V$$

where ΔV is the final value of the stress-free volume change. Show that it is also given in terms of the actual final (static) volume change δV by

$$M_0(\infty) = (\lambda + 2\mu)\ \delta V.$$

d) Show that the actual volume change δV is independent of a, in the sense that having proved $M_0(\infty) = (\lambda + 2\mu)\delta V$ as above for some small value of a, we can choose a larger value of a and evaluate the outward actual static displacement for this new surface. But δV is unchanged in value. (*Hint:* Use the exterior solution, mentioned in Problem 3.7.)

[We shall find in later chapters that the time-dependent moment is an important property of the seismic source, which can often be obtained from seismograms, and that the final value $M_0(\infty)$ is related simply to the long-period spectrum of observed signals. Our equations relating moment to ΔV and δV enable measurements of $M_0(\infty)$ to be interpreted in terms of volume change at the source, for isotropic sources. We see that the actual volume change of a source that wants to expand, δV in the present problem, is approximately half the size of the stress-free volume change (since $(\lambda + 2\mu)/(\lambda + \frac{2}{3}\mu) \sim 2$), as noted by Müller (2001). Earlier, Müller (1973b) showed that for isotropic sources the scalar moment is $(\lambda + 2\mu) \times$ area $\times$ outward displacement. In the notation used here, area $\times$ outward displacement $= 4\pi a^2 \times \delta a = \delta V$. Explosive sources are sometimes quantified by this volume change. We are free to take the value of a large enough to confine all nonelastic processes to the interior region, and as noted above the actual volume increment δV has meaning independent of any value of a. The actual volume increment δV represents the expansion that the nonlinear source region applies to the external linearly elastic region. Müller's 1973 result complements the fact that for shear faulting the double couple is based on a scalar moment given by $\mu \times$ area $\times$ slip. This too can be thought of as the output from the nonlinear region, where rocks are fracturing and shearing, applied to the external elastic region. The product given by area $\times$ slip is called the *potency*. Heaton and Heaton (1989) and Ben-Zion (2001) recommend that the potency be used to quantify earthquake (shear dislocation) sources, instead of the seismic moment ($\mu \times$ potency). Like δV, potency has the dimensions of volume change. The potency, δV, and seismic moment all provide ways to characterize quantitative attributes of the nonlinear source, which are needed to interpret measurements made in the linearly elastic region.]