CHAPTER 2

Basic Theorems in Dynamic Elasticity

An analytical framework for studying seismic motions in the Earth must incorporate, at the very least, the following three components: a description of seismic sources, equations for the motions that can propagate once motion has somewhere been initiated, and a theory coupling the source description into the particular solution sought for the equations of motion. It will be useful if the theory can be simplified by taking full advantage of our conjectures about seismic motion (though such a theory may mislead the user if the conjectures are invalid). For example, there is the conjecture that two sets of small motions may be superimposed without interfering with each other. Another conjecture is that the seismic motions set up by some physical source should be uniquely determined by the combined properties of that source and the medium in which the waves propagate. These conjectures, and many others that are generally assumed by seismologists to be true, are properties of infinitesimal motion in classical continuum mechanics for an elastic medium with a linear stress–strain relation; such a theory will provide the mathematical framework for almost all of this text.

Seismology is largely an observational science, so the ability to interpret seismograms is fundamental to progress. For this reason, there is a need to know what information about the motion in one part of a medium is enough to determine uniquely the motion that may be observed in another part. As a practical example, we often need to know how to characterize a seismic source (an explosion or a spontaneous fault motion) and how to allow for boundary conditions at the Earth's free surface in order to determine the resulting motion at a network of receivers. Fortunately, for a linear elastic medium, this problem has a definite solution, in that prescribed source conditions (in terms of body forces) and boundary conditions can readily be stated in forms that do enforce uniqueness for the resulting motions. After giving a formulation of the problem (i.e., establishing notation; defining displacement, strain, traction, body force, and stress; and stating constraints on the motion), we prove the two fundamental theorems of uniqueness and reciprocity. Reciprocity is used together with a Green function to obtain a representation of motion at a general point in the medium in terms of body forces and information on boundaries. This method of representation in elastodynamics is due to Knopoff (1956) and de Hoop (1958). It has many familiar parallels in complex number theory, in potential theory, and in the theory of the scalar wave equation for a homogeneous medium.

BOX **2.1**

Examples of representation theorems

1. If $f(z)$ is an analytic function of the complex variable z, then

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta) d\zeta}{\zeta - z},$$

where the contour integral is taken counterclockwise on any path C around the point z. (No singularities of f are allowed inside C.) This formula is then a *representation* of the function f, which allows f to be evaluated everywhere inside C provided the values of f are known on C itself.

2. If $\phi(x, y, z)$ satisfies the Poisson equation $\nabla^2\phi = -4\pi\rho$, then

$$\phi(\mathbf{x}) = \iiint_V \frac{\rho(\xi)\, dV(\xi)}{|\mathbf{x} - \xi|},$$

where V is a volume including all of the density distribution ρ that contributes to ϕ. This too is a *representation* of ϕ, but one that does not involve values of ϕ itself.

The elastodynamic representation theorem involves both the above types of representation, and also incorporates time dependence.

It is often useful to have the equations of elastic motion referred to general orthogonal curvilinear coordinate systems, since, in many instances, the (curved) coordinate surfaces are just those on which it is natural to apply a boundary condition. We derive the displacement–stress equations and the strain–displacement equations, using the physical components of displacement, stress, and strain in a general orthogonal system.

This chapter may seem at first sight to consist mainly of formal results—of proofs that must be established once, by one person, to legitimize the specific problem-solving methods expounded in later chapters. However, the reader who wishes to develop the ability to solve problems in theoretical or applied seismology on his or her own will soon face the question of how a problem is "set up." That is, how does one translate the physical description of a seismic source—and the general problem of calculating the ensuing motions at nearby and/or distant receivers—into a specific mathematical problem? In large part, the ability to set up such problems will stem from mastery of the representation theorem, given in various forms by equations (2.41)–(2.43) and (3.1)–(3.3). We shall frequently refer to these equations in later chapters.

2.1 Formulation

Two different methods are widely used to describe the motions and the mechanics of motion in a continuum. These are the Lagrangian description, which emphasizes the study of a particular particle that is specified by its original position at some reference time, and the Eulerian description, which emphasizes the study of whatever particle happens to occupy a particular spatial location. For most applications in seismology, the linear theory of elasticity is conceptually simpler to develop with the Lagrangian description, and this is the framework we shall almost always adopt. Note that a seismogram is the record of motion of a particular part of the Earth (namely, the particles to which the seismometer was attached during installation), so it is directly a record of Lagrangian motion.

We shall work in this chapter with a Cartesian coordinate system (x_1, x_2, x_3), and all tensors here are Cartesian tensors. We use the term *displacement*, regarded as a function of space and time, and written as $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, to denote the vector distance of a particle at time t from the position $\mathbf{x}$ that it occupies at some reference time t_0, often taken as $t = 0$. Since $\mathbf{x}$ does not change with time, it follows that the *particle velocity* is $\partial\mathbf{u}/\partial t$ and that the *particle acceleration* is $\partial^2\mathbf{u}/\partial t^2$.

To analyze the distortion of a medium, whether it be solid or fluid, elastic or inelastic, we use the *strain tensor*. If a particle initially at position $\mathbf{x}$ is moved to position $\mathbf{x} + \mathbf{u}$, then the relation $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is used to describe the displacement field. To examine the distortion of the part of the medium that was initially in the vicinity of $\mathbf{x}$, we need to know the new position of the particle that was initially at $\mathbf{x} + \delta\mathbf{x}$. This new position is $\mathbf{x} + \delta\mathbf{x} + \mathbf{u}(\mathbf{x} + \delta\mathbf{x})$. Any distortion is liable to change the relative position of the ends of the line-element $\delta\mathbf{x}$. If this change is $\delta\mathbf{u}$, then $\delta\mathbf{x} + \delta\mathbf{u}$ is the new vector line-element, and by writing down the difference between its end points we obtain

$$\delta\mathbf{x} + \delta\mathbf{u} = \mathbf{x} + \delta\mathbf{x} + \mathbf{u}(\mathbf{x} + \delta\mathbf{x}) - (\mathbf{x} + \mathbf{u}(\mathbf{x})).$$

Since $|\delta\mathbf{x}|$ is arbitrarily small, we can expand $\mathbf{u}(\mathbf{x} + \delta\mathbf{x})$ as $\mathbf{u} + (\delta\mathbf{x} \cdot \nabla)\mathbf{u}$ plus negligible terms of order $|\delta\mathbf{x}|^2$. It follows that $\delta\mathbf{u}$ is related to gradients of $\mathbf{u}$ and to the original line-element $\delta\mathbf{x}$ via

$$\delta\mathbf{u} = (\delta\mathbf{x} \cdot \nabla)\mathbf{u}, \qquad \text{or} \qquad \delta u_i = \frac{\partial u_i}{\partial x_j}\,\delta x_j. \tag{2.1}$$

However, we do not need all of the nine independent components of the tensor u_{ij} to specify true distortion in the vicinity of $\mathbf{x}$, since part of the motion is due merely to an infinitesimal rigid-body rotation of the neighborhood of $\mathbf{x}$. This can be seen from the identity $(u_{i,j} - u_{j,i})\delta x_j = \varepsilon_{ijk}\varepsilon_{jlm}u_{m,l}\delta x_k$ (see Box 2.2 and Problem 2.2), so that equation (2.1) can be rewritten as

$$\delta u_i = \tfrac{1}{2}(u_{i,j} + u_{j,i})\delta x_j + \tfrac{1}{2}(\text{curl } \mathbf{u} \times \delta\mathbf{x})_i, \tag{2.2}$$

and the rigid-body rotation is of amount $\frac{1}{2}\text{curl } \mathbf{u}$. The interpretation of the last term in (2.2) as a rigid-body rotation is valid if $\left|u_{i,j}\right| \ll 1$. If displacement gradients were not "infinitesimal" in the sense of this inequality, then we should instead have to analyze the contribution to $\delta\mathbf{u}$ from a *finite* rotation—a much more difficult matter, since finite rotations do not commute and cannot be expressed as vectors.

In terms of the infinitesimal strain tensor, defined to have components

$$e_{ij} \equiv \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.3}$$

BOX **2.2**

Notation

We shall use boldface symbols (e.g., $\mathbf{u}$, $\boldsymbol{\tau}$) for vector and tensor fields, and subscripts (e.g., u_i, τ_{kl}) to designate vector and tensor components in a Cartesian coordinate system. Useful references for the properties of Cartesian tensors are Jeffreys (1965) and Chapter 3 of Jeffreys and Jeffreys (1972).

For unit vectors (other than $\boldsymbol{\nu}$, $\mathbf{l}$, $\mathbf{n}$, $\mathbf{b}$), the circumflex is used (e.g., $\hat{\mathbf{x}}$). Scalar products are written as $\mathbf{a} \cdot \mathbf{b}$, and vector products are written as $\mathbf{a} \times \mathbf{b}$.

Overdots are used to indicate time derivatives (e.g., $\dot{\mathbf{u}} = \partial \mathbf{u}/\partial t$, $\ddot{\mathbf{u}} = \partial^2 \mathbf{u}/\partial t^2$), and a comma between subscripts is used for spatial derivatives (e.g., $u_{i,j} = \partial u_i / \partial x_j$).

The summation convention for repeated subscripts is followed throughout (e.g., $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = \mathbf{a} \cdot \mathbf{b}$), and frequent use is made of the Kronecker symbol δ_{ij} and the alternating tensor with components ε_{ijk}:

$$\delta_{ij} = 0 \quad \text{for} \quad i \neq j, \quad \text{and} \quad \delta_{ij} = 1 \quad \text{for} \quad i = j;$$

$$\varepsilon_{ijk} = 0 \quad \text{if any of } i, j, k \text{ are equal,}$$

otherwise

$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = -\varepsilon_{213} = -\varepsilon_{321} = -\varepsilon_{132} = 1.$$

The most important properties of these symbols are then

$$a_i = \delta_{ij} a_j, \qquad \varepsilon_{ijk} a_j b_k = (\mathbf{a} \times \mathbf{b})_i;$$

and they are linked by the properties

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \quad \text{and} \quad \varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{jl} & \delta_{kl} \\ \delta_{im} & \delta_{jm} & \delta_{km} \\ \delta_{in} & \delta_{jn} & \delta_{kn} \end{vmatrix}.$$

The second-order tensor $\mathbf{t}$ is symmetric if and only if $\varepsilon_{ijk} t_{jk} = 0$.

the effect of true distortion on any line-element δx_i is to change the relative position of its end points by $e_{ij}\delta x_j$. Rotation does not affect the length of the element, and the new length is

$$\begin{aligned} |\delta \mathbf{x} + \delta \mathbf{u}| &= \sqrt{\delta \mathbf{x} \cdot \delta \mathbf{x} + 2\delta \mathbf{u} \cdot \delta \mathbf{x}} && \text{(neglecting } \delta \mathbf{u} \cdot \delta \mathbf{u}) \\ &= \sqrt{\delta x_i \delta x_i + 2 e_{ij} \delta x_i \delta x_j} && \text{(from (2.2), and using (curl } \mathbf{u} \times \delta \mathbf{x}) \cdot \delta \mathbf{x} = 0) \\ &= |\delta \mathbf{x}| \, (1 + e_{ij} \nu_i \nu_j) && \text{(to first order, if } |\, e_{ij} \,| \ll 1), \end{aligned}$$

where $\boldsymbol{\nu}$ is the unit vector $\delta \mathbf{x} / |\delta \mathbf{x}|$. It follows that the extensional strain of a line-element originally in the $\boldsymbol{\nu}$ direction is $e_{ij} \nu_i \nu_j$.

To analyze the internal forces acting mutually between adjacent particles within a continuum, we use the concepts of *traction* and *stress tensor*. Traction is a vector, being the

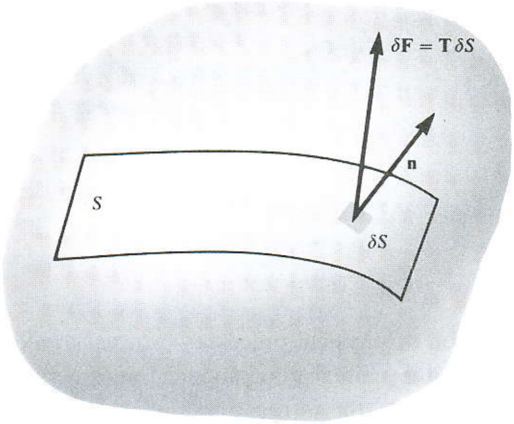

FIGURE 2.1
The definition of traction **T** acting at a point across the internal surface S with normal **n**. The choice of sign is such that traction is a pulling force. Pushing is in the opposite direction, so for a fluid medium, the pressure would be $-\mathbf{n} \cdot \mathbf{T}$.

force acting per unit area across an internal surface within the continuum, and quantifies the contact force (per unit area) with which particles on one side of the surface act upon particles on the other side. For a given point of the internal surface, traction is defined (see Fig. 2.1) by considering the infinitesimal force $\delta\mathbf{F}$ acting across an infinitesimal area δS of the surface, and taking the limit of $\delta\mathbf{F}/\delta S$ as $\delta S \to 0$. With a unit normal **n** to the surface S, the convention is adopted that $\delta\mathbf{F}$ has the direction of force due to material on the side to which **n** points and acting upon material on the side from which **n** is pointing; the resulting traction is denoted as $\mathbf{T}(\mathbf{n})$. If $\delta\mathbf{F}$ acts in the direction shown in Fig. 2.1, traction is a pulling force, opposite to a pushing force such as pressure. Thus, in a fluid, the (scalar) pressure is $-\mathbf{n} \cdot \mathbf{T}(\mathbf{n})$. For a solid, shearing forces can act across internal surfaces, and so **T** need not be parallel to **n**. Furthermore, the magnitude and direction of traction depend on the orientation of the surface element δS across which contact forces are taken (whereas pressure at a point in a fluid is the same in all directions). To appreciate this orientation-dependence of traction at a point, consider a point P, as shown in Figure 2.2, on the exterior surface of a house. For an element of area on the surface of the wall at P, the traction $\mathbf{T}(\mathbf{n}_1)$ is zero (neglecting atmospheric pressure and winds); but for a horizontal element of area within the wall at P, the traction $\mathbf{T}(\mathbf{n}_2)$ may be large (and negative).

The forces acting upon particles in a solid or fluid medium consist not only of the contact forces between adjacent particles, but also of (i) forces between particles that are

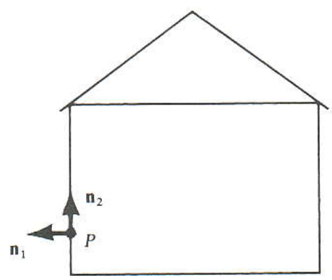

FIGURE 2.2
$\mathbf{T}(\mathbf{n}_1) \neq \mathbf{T}(\mathbf{n}_2)$.

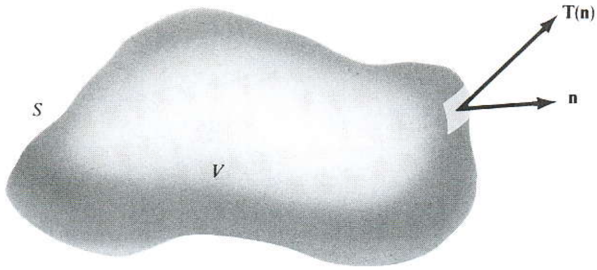

FIGURE **2.3**
A material volume V of the continuum, with surface S.

not adjacent, and (ii) forces due to the application of physical processes external to the medium itself. An example of type (i) would be the mutual gravitational forces acting between particles of the Earth. Type (ii) is illustrated by the forces on buried particles of iron when a magnet is moved around outside the medium in which the iron is contained. To these noncontact forces, we give the name *body forces*, and use the notation $\mathbf{f}(\mathbf{x}, t)$ to denote the body force acting per unit volume on the particle originally at position $\mathbf{x}$ at some reference time. It will often be useful to consider the special case of a force applied impulsively to one particular particle at $\mathbf{x} = \xi$ and time $t = \tau$. If this force is in the direction of the x_n-axis, it follows that $f_i(\mathbf{x}, t)$ is proportional to the three-dimensional Dirac delta function $\delta(\mathbf{x} - \xi)$, specifying the spatial location; to the one-dimensional Dirac delta function $\delta(t - \tau)$, specifying the timing of the impulse; and to the Kronecker delta function δ_{in}, signifying the directional property that $f_i = 0$ for $i \neq n$. Thus the body-force distribution in this case is given by

$$f_i(\mathbf{x}, t) = A\, \delta(\mathbf{x} - \xi)\, \delta(t - \tau)\, \delta_{in}, \tag{2.4}$$

where A is a constant giving the strength of the impulse. Note that the dimensions of f_i, $\delta(\mathbf{x} - \xi)$, and $\delta(t - \tau)$ are, respectively, force per unit volume, 1/unit volume, and 1/unit time. The Kronecker delta is dimensionless, so A does have the correct physical dimension for an impulse (force × time).

We are now in a position to place a constraint on the accelerations, body forces, and tractions acting throughout a volume V with surface S (see Fig. 2.3). By equating the rate of change of momentum of particles constituting V to the forces acting on these particles, we find

$$\frac{\partial}{\partial t} \iiint_V \rho \frac{\partial \mathbf{u}}{\partial t}\, dV = \iiint_V \mathbf{f}\, dV + \iint_S \mathbf{T}(\mathbf{n})\, dS. \tag{2.5}$$

This relation is based on a Lagrangian description, and V and S move with the particles. The left-hand side can thus be written as $\iiint_V \rho(\partial^2\mathbf{u}/\partial t^2)\, dV$, since the particle mass $\rho\, dV$ is constant in time.

Our first use of (2.5) is to obtain an explicit form for the functional relationship $\mathbf{T} = \mathbf{T}(\mathbf{n})$ and to introduce the stress tensor. Consider a particle P within the medium for which the acceleration, the body force, and the tractions are all nonsingular. Surround this particle by a small volume ΔV, and consider the relative magnitude of the three terms in

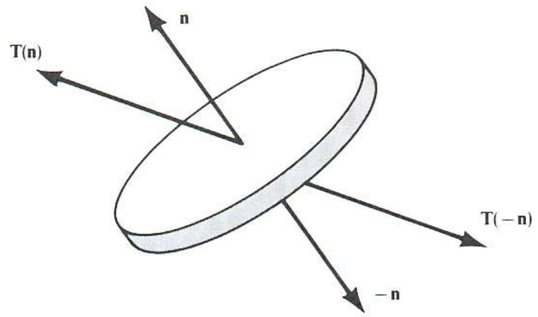

FIGURE 2.4
A small disc within a stressed medium.

(2.5) as ΔV shrinks down onto P. The volume integrals will be of order ΔV, but the surface integral is of order $\iint_S dS$ taken over the surface of ΔV. In general such integrals are of order $(\Delta V)^{2/3}$, tending to zero more slowly than ΔV. After dividing (2.5) through by $\iint_S dS$, it follows that

$$\frac{\left|\iint \mathbf{T}\, dS\right|}{\iint dS} = O(\Delta V^{1/3}) \to 0 \quad \text{as} \quad \Delta V \to 0. \tag{2.6}$$

Now suppose that ΔV is a disc, with opposite faces having outward normals $\mathbf{n}$ and $-\mathbf{n}$ (see Fig. 2.4) and the edge having insignificant area. Equation (2.6) then implies the result

$$\mathbf{T}(-\mathbf{n}) = -\mathbf{T}(\mathbf{n}). \tag{2.7}$$

Next, take ΔV to be a small tetrahedron, with three of its faces in the coordinate planes (see Fig. 2.5) and the fourth having $\mathbf{n}$ as its outward normal. Equation (2.6) then implies

$$\frac{\mathbf{T}(\mathbf{n})ABC + \mathbf{T}(-\hat{\mathbf{x}}_1)OBC + \mathbf{T}(-\hat{\mathbf{x}}_2)OCA + \mathbf{T}(-\hat{\mathbf{x}}_3)OAB}{ABC + OBC + OCA + OAB} \to \mathbf{0} \tag{2.8}$$

as $\Delta V \to 0$. Here, the symbols ABC etc. denote areas of triangles, and one can show geometrically that the components of $\mathbf{n}$ are given by $(n_1, n_2, n_3) = (OBC, OCA, OAB)/ABC$. Then (2.8) and (2.7) yield

$$\mathbf{T}(\mathbf{n}) = \mathbf{T}(\hat{\mathbf{x}}_j)n_j, \tag{2.9}$$

which is a specific and important relationship between traction $\mathbf{T}(\mathbf{n})$ and $\mathbf{n}$ in terms of three tractions acting across coordinate planes. The properties (2.7) and (2.9) are trivial for a static medium, but we have shown them to be true even during acceleration.

The stress tensor is introduced by defining the nine quantities

$$\tau_{kl} = T_l(\hat{\mathbf{x}}_k),$$

so that τ_{kl} is the lth component of the traction acting across the plane normal to the kth axis due to material with greater x_k acting upon material with lesser x_k. Thus

$$T_i = \tau_{ji}n_j. \tag{2.10}$$

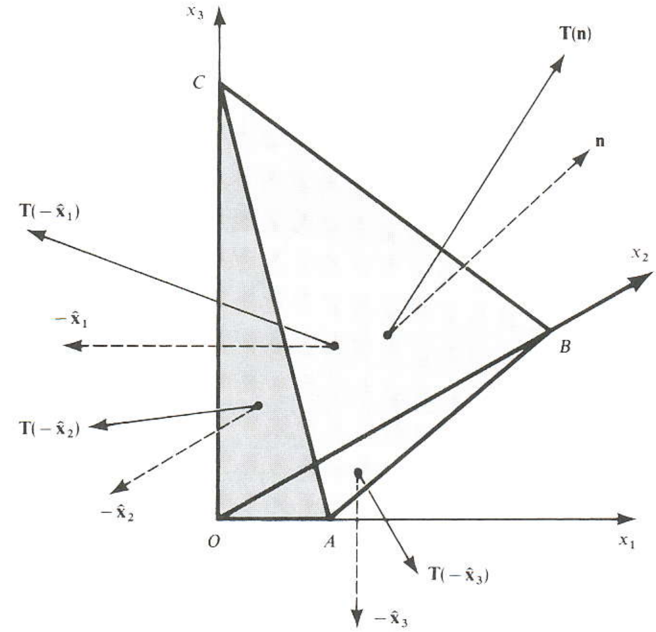

FIGURE **2.5**
The small tetrahedron *OABC* has three of its faces in the coordinate planes, with outward normals $-\hat{\mathbf{x}}_j (j = 1, 2, 3)$, and the fourth face has normal $\mathbf{n}$.

Our second use of (2.5) is to obtain the equation of motion of a general particle. Applying (2.10) and Gauss's divergence theorem to give

$$\iint_S T_i \, dS = \iint_S \tau_{ji} n_j \, dS = \iiint_V \tau_{ji,j} \, dV, \tag{2.11}$$

we find for a general volume V that

$$\iiint_V (\rho \ddot{u}_i - f_i - \tau_{ji,j}) \, dV = 0. \tag{2.12}$$

This integrand must be zero wherever it is continuous, otherwise a volume V could be found that violates (2.12), hence

$$\rho \ddot{u}_i = f_i + \tau_{ji,j}, \tag{2.13}$$

which is our first form for the equation of motion.

Another constraint upon the mechanics of motion is given by equating the rate of change of angular momentum about the origin of coordinates to the moment of forces acting on the particles in V. Thus

$$\frac{\partial}{\partial t} \iiint_V \mathbf{X} \times \rho \dot{\mathbf{u}} \, dV = \iiint_V \mathbf{X} \times \mathbf{f} \, dV + \iint_S \mathbf{X} \times \mathbf{T} \, dS, \tag{2.14}$$

where $\mathbf{X} = \mathbf{x} + \mathbf{u}$. Since $\partial \mathbf{x}/\partial t$, $\dot{\mathbf{u}} \times \dot{\mathbf{u}}$, and $\partial(\rho \, dV)/\partial t$ are all zero, the left-hand side here is $\iiint_V \mathbf{X} \times \rho \ddot{\mathbf{u}} \, dV$. Using the strict interpretation of (2.13) developed in Box 2.3,

BOX **2.3**

Euler or Lagrange?

A closer look at the application of Gauss's theorem in (2.11) shows that our Lagrangian approach is inappropriate for the spatial differentiations in (2.11)–(2.13). The particles constituting S at time t have, in general, moved from their position at the reference time t_0, so that

$$\iint_S \tau_{ji} n_j \, dS = \iiint_V \frac{\partial}{\partial X_j} \tau_{ji} \, dV,$$

where $\mathbf{X} = \mathbf{x} + \mathbf{u}$, and the spatial differentiation that must be conducted on points throughout V at time t is $\partial/\partial X_j$. For finite motions, the exact equation for motion in the continuum is therefore, in our notation,

$$\rho \frac{\partial^2 u_i}{\partial t^2} = f_i + \frac{\partial \tau_{ji}}{\partial X_j}. \qquad \text{(2.13, strict form)}$$

The Eulerian approach instead discusses field variables directly as a function of $\mathbf{X}$ and t (taking $\mathbf{u}$ to be the displacement of the particle at $\mathbf{X}$ and time t from its position $\mathbf{x}$ at time t_0), and τ_{ji} would be a stress component at $(\mathbf{X}, t)$. This offers the advantage of allowing one to work with independent variables that are natural for interpreting the right-hand side of the equation of motion, but has the disadvantage of cumbersome expressions for the rate of change of properties carried by the particles. For example, particle velocity $\mathbf{v}$ at $(\mathbf{X}, t)$ is difficult to express in terms of the displacement field $\mathbf{u} = \mathbf{u}(\mathbf{X}, t)$. The equation for $\mathbf{v}$ is given by seeing that the particle at $\mathbf{X}$ at time t has moved to $\mathbf{X} + \delta\mathbf{X}$ at time $t + \delta t$, so

$$\mathbf{v}\,\delta t = \mathbf{u}(\mathbf{X} + \delta\mathbf{X}, t + \delta t) - \mathbf{u}(\mathbf{X}, t).$$

Since $\mathbf{v}$ = limit of $\delta\mathbf{X}/\delta t$ for a fixed particle,

$$v_i = \left(\frac{\partial u_i}{\partial t}\right)_{\text{fixed position}} + v_j \left(\frac{\partial u_i}{\partial X_j}\right)_{\text{fixed time}}$$

is the implicit equation to be solved for $\mathbf{v}$ in terms of $\mathbf{u}$ (implicit, because components of $\mathbf{v}$ appear on both sides of the equation). Once the particle velocity is found, the acceleration of the particle at $(\mathbf{X}, t)$ is easily given by the material derivative $\partial\mathbf{v}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{v}$, where ∇ is the Eulerian spatial derivative, i.e., in $\mathbf{X}$ coordinates.

In seismology, the distinction between Lagrangian and Eulerian approaches rarely needs to be made, since spatial fluctuations in the displacements, strains, accelerations, and stresses have wavelengths much greater than the amplitude of particle displacements. In this case, it makes no practical difference whether a spatial gradient is evaluated at a fixed position (Euler) or for a particular particle (Lagrange). In this book we emphasize the Lagrangian approach because there is then a simple exact relationship between particle velocity and particle displacement, $\mathbf{v} = \partial\mathbf{u}(\mathbf{x}, t)/\partial t$, and because a seismometer measures the motion of the fixed particles to which it was originally attached. In fluid mechanics, where particle displacements may not be small, there is little interest in particle displacement as a field variable and the Eulerian approach is more useful.

A final acknowledgment: the "Eulerian" and "Lagrangian" approaches were both developed by Leonhard Euler.

it follows that

$$\iiint_V \varepsilon_{ijk} X_j \frac{\partial}{\partial X_l} \tau_{lk}\, dV = \iiint_V \varepsilon_{ijk} X_j (\rho \ddot{u}_k - f_k)\, dV$$

$$= \iint_S \varepsilon_{ijk} X_j T_k\, dS \qquad \text{(from (2.14))}$$

$$= \iint_S \varepsilon_{ijk} X_j \tau_{lk} n_l\, dS \qquad \text{(from (2.10))}.$$

Applying the divergence theorem to this surface integral and using $\partial X_j / \partial X_l = \delta_{jl}$, one obtains

$$\iiint_V \varepsilon_{ijk} \tau_{jk}\, dV = 0 \qquad \text{for any volume } V,$$

implying $\varepsilon_{ijk} \tau_{jk} = 0$ everywhere, and hence that the stress tensor is symmetric:

$$\tau_{kj} = \tau_{jk}. \tag{2.15}$$

With this fundamental result, we can finally state the formula for traction components as

$$T_i = \tau_{ij} n_j \tag{2.16}$$

and the equation of motion as

$$\rho \ddot{u}_i = f_i + \tau_{ij,j}. \tag{2.17}$$

The spatial derivative here should be carried out with respect to X_j, but (as discussed in Box 2.3) differentiation with respect to x_j is usually adequate in seismology, and will henceforth be assumed.

2.2 Stress–Strain Relations and the Strain–Energy Function

A medium is said to be *elastic* if it possesses a natural state (in which strains and stresses are zero) to which it will revert when applied forces are removed. Under the influence of applied loads, stress and strain will change together, and the relation between them, called the constitutive relation, is an important characteristic of the medium. That there is such a relation we prove below by thermodynamic arguments. The relation itself is a proper subject for experimental determination, and Robert Hooke's measurements of "springy bodies" led him, over 300 years ago, to the conclusion that stress is proportional to strain. His statements on this matter were somewhat enigmatic, as today's concepts of traction and tensor were then unavailable. Augustin Cauchy, in the early nineteenth century, was the first to develop many of our modern ideas of traction, and it is clear that he understood many results that today are more easily communicated in terms of tensors, which did not come into general use until the twentieth century. The modern generalization of Hooke's law is that each component of the stress tensor is a linear combination of all components of the strain tensor, i.e., that there exist constants c_{ijkl} such that

$$\tau_{ij} = c_{ijpq} e_{pq}. \tag{2.18}$$

A body that obeys the constitutive relation (2.18) is said to be *linearly elastic*. The quantities c_{ijkl} are components of a fourth-order tensor, and have the symmetries

$$c_{jipq} = c_{ijpq} \qquad (\text{due to } \tau_{ji} = \tau_{ij}) \tag{2.19}$$

and

$$c_{ijqp} = c_{ijpq} \qquad (\text{due to } e_{qp} = e_{pq}). \tag{2.20}$$

It is also true from a thermodynamic argument that $c_{pqij} = c_{ijpq}$, as we now shall show.

Suppose that an elastic body occupies the volume V with surface S. The first law of thermodynamics states that the body possesses an internal (or, intrinsic) energy, which may change with deformations of the body, and the energy balance for work done on the body is:

$$\begin{aligned}&\text{Rate of doing mechanical work} + \text{Rate of heating}\\&\quad = \text{Rate of increase of (kinetic + internal energies).}\end{aligned} \tag{2.21}$$

Let us analyze each of these terms separately.

(1) The rate of mechanical work is given by

$$\begin{aligned}&\iiint_V \mathbf{f}\cdot\dot{\mathbf{u}}\,dV + \iint_S \mathbf{T}\cdot\dot{\mathbf{u}}\,dS\\
&= \iiint_V \left[f_i\dot{u}_i + (\tau_{ij}\dot{u}_i)_{,j}\right]\,dV \qquad (\text{from (2.16) and Gauss's divergence theorem})\\
&= \iiint_V (\rho\dot{u}_i\ddot{u}_i + \tau_{ij}\dot{u}_{i,j})\,dV \qquad (\text{from (2.17)})\\
&= \frac{\partial}{\partial t}\iiint_V \frac{1}{2}\rho\dot{u}_i\dot{u}_i\,dV + \iiint_V \tau_{ij}\dot{e}_{ij}\,dV \quad (\text{symmetry of } \tau_{ij} \text{ and definition of } e_{ij}).\end{aligned} \tag{2.22}$$

(2) Let $\mathbf{h}(\mathbf{x}, t)$ be the heat flux, such that $\mathbf{h}\cdot\mathbf{n}$ is the rate at which heat is transmitted (per unit area) in the $\mathbf{n}$ direction across area elements normal to $\mathbf{n}$. Let $\mathcal{Q}(\mathbf{x}, t)$ be the heat per unit volume due to input through the boundary, so that the rate of heating is given by

$$\frac{\partial}{\partial t}\iiint_V \mathcal{Q}\,dV = -\iint_S \mathbf{h}\cdot\mathbf{n}\,dS. \tag{2.23}$$

Then clearly $\dot{\mathcal{Q}} = -\nabla\cdot\mathbf{h}$.

(3) The rate of increase of kinetic energy is given by

$$\frac{\partial}{\partial t}\iiint_V \frac{1}{2}\rho\dot{u}_i\dot{u}_i\,dV. \tag{2.24}$$

(4) Let $\mathcal{U}$ be the internal energy per unit volume. Then from (2.21)–(2.24) we conclude that

$$\dot{\mathcal{U}} = -h_{i,i} + \tau_{ij}\dot{e}_{ij}, \qquad \text{or} \qquad \dot{\mathcal{U}} = \dot{\mathcal{Q}} + \tau_{ij}\dot{e}_{ij}. \tag{2.25}$$

If $\mathcal{U}$, $\mathcal{Q}$, and e_{ij} are measured as small perturbations away from a state of thermodynamic equilibrium, then (2.25) is equivalent to

$$\begin{aligned} d\mathcal{U} &= d\mathcal{Q} + \tau_{ij}\, de_{ij} \\ &= \mathcal{T}\, d\mathcal{S} + \tau_{ij}\, de_{ij} \qquad \text{(for reversible processes),} \end{aligned} \tag{2.26}$$

in which $\mathcal{S}$ is the entropy per unit volume and $\mathcal{T}$ is the absolute temperature. Equation (2.26) implies that entropy and strain components are the state variables in terms of which internal energy is completely and uniquely specified. In particular, internal energy does not depend on the time history of strain.

It is often useful to work with a function $\mathcal{W}$ of the strain components that allows the stresses to be generated via

$$\tau_{ij} = \frac{\partial \mathcal{W}}{\partial e_{ij}}. \tag{2.27}$$

A function with this property is called a *strain–energy function*.

Note from (2.26) the formal result

$$\tau_{ij} = \left(\frac{\partial \mathcal{U}}{\partial e_{ij}}\right)_{\mathcal{S}}. \tag{2.28}$$

If the processes of deformation are adiabatic, so that $\mathbf{h} = \mathbf{0}$ and $\dot{\mathcal{Q}} = 0$, then the actual changes in $\mathcal{U}$ associated with changes in strain do occur at constant entropy, and we can choose $\mathcal{W} = \mathcal{U}$ and use (2.28). This is the situation in seismology, since the time constant of thermal diffusion in rock ((distance)^2/diffusivity) is very much longer than the period of seismic waves (wavelength/velocity).

It is also true that $\tau_{ij} = (\partial\mathcal{F}/\partial e_{ij})_{\mathcal{T}}$, where $\mathcal{F} = \mathcal{U} - \mathcal{T}\mathcal{S}$ is the *free energy* per unit volume (for which $d\mathcal{F} = -\mathcal{S}\, d\mathcal{T} + \tau_{ij}\, de_{ij}$). For deformation processes that take place so slowly as to be isothermal, as in some tectonic processes, it is then natural to form τ_{ij} from changes in the free energy, and one would choose $\mathcal{W} = \mathcal{F}$.

For all deformations such that the strain–energy function exists, we may combine its properties with Hooke's law and find

$$\frac{\partial \mathcal{W}}{\partial e_{ij}} = \tau_{ij} = c_{ijpq}e_{pq}, \tag{2.29}$$

which implies

$$c_{pqij} = c_{ijpq} \qquad \left(\text{from} \qquad \frac{\partial^2 \mathcal{W}}{\partial e_{ij}\partial e_{pq}} = \frac{\partial^2 \mathcal{W}}{\partial e_{pq}\partial e_{ij}}\right). \tag{2.30}$$

Since all the first derivatives of $\mathcal{W}$ are homogeneous (of order one) in strain components, and $\mathcal{W}$ can be taken as zero in the natural state, $\mathcal{W}$ itself must be homogeneous (of order two) in the form

$$\mathcal{W} = d_{ijpq}e_{ij}e_{pq}. \tag{2.31}$$

This quadratic is the same as $\frac{1}{2}(d_{ijpq} + d_{pqij})e_{ij}e_{pq}$, but differentiation of (2.31) to give τ_{ij} shows that $(d_{ijpq} + d_{pqij}) = c_{ijpq}$, hence the strain–energy function is, explicitly,

$$\mathcal{W} = \tfrac{1}{2}c_{ijkl}e_{ij}e_{kl} = \tfrac{1}{2}\tau_{ij}e_{ij}. \tag{2.32}$$

Under adiabatic or isothermal conditions, the strain–energy function is positive except for the natural state (where $\mathcal{W} = 0$), so that $\frac{1}{2}c_{ijkl}e_{ij}e_{kl}$ is a positive definite quadratic form. ($\mathcal{W} \geq 0$, because we assume the natural state is stable.)

The c_{ijkl} are independent of strain, which is why they are sometimes called "elastic constants," although they are varying functions of position in the Earth. The elasticity theory used in seismology is to a large extent characterized by a preoccupation with inhomogeneous media, particularly with a spherically symmetric medium that is everywhere isotropic. In general, the symmetries (2.19), (2.20), and (2.30) reduce the number of independent components in c_{ijkl} from 81 to 21. There is considerable simplification in the case of an isotropic medium, since **c** must be isotropic. It can be shown (Jeffreys and Jeffreys, 1972) that the most general isotropic fourth-order tensor, having the symmetries of **c**, has the form

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2.33}$$

This involves only two independent constants, λ and μ, known as the Lamé moduli.

Note that the results we have obtained in the present section are specialized to the case of small perturbations away from a reference state in which strain and stress are both zero. In the Earth's interior, self-gravitation is responsible for pressures of up to around 1 megabar. Even if one postulates a state of zero stress and strain for Earth materials, it is clear that the results of this section cannot directly be applied in seismology, since strains due to such pressures are not small. Using such a reference state, one must work with a theory of finite strain, in which the stress–strain relation is nonlinear. Alternatively, one can choose the static equilibrium configuration of the Earth, prior to an earthquake, as a reference state. This is the usual procedure in seismology. By definition, the reference state is one of zero strain, but now the initial stress is nonzero, and seismic motions are studied in terms of a linear relationship between strains and *incremental stresses*. Thus the stress is $\boldsymbol{\sigma}^0$ at zero strain, and is $\boldsymbol{\sigma}^0 + \boldsymbol{\tau}$ at nonzero strain, where $\tau_{ij} = c_{ijkl}e_{kl}$, and components σ^0_{ij} can be of the same order as components c_{ijkl} ($\sim$ 1 megabar).

For the present, we shall continue to neglect the effects of initial stress $\boldsymbol{\sigma}^0$. This simplification is justified in Chapter 8, where initial stresses are correctly taken into account and where a brief review is given of those aspects of the theory that need revision (Box 8.5). To quantify the effects of self-gravitation, we shall in Chapter 8 adopt an Eulerian approach.

2.3 Theorems of Uniqueness and Reciprocity

It is natural to introduce the discussion of uniqueness (for the displacement field **u** throughout a body with volume V and surface S) with some general remarks concerning the ways in which motion can be set up. Because the displacement is constrained to satisfy (2.17) throughout V, the application of body forces will generate a displacement field, as will the application of tractions on the surface S. We shall show that specification of the body forces throughout V, and tractions over all of S, is enough to determine uniquely the displacement field that will develop throughout V from given initial conditions. An alternative way to specify the influence of S on the displacement field is to give a boundary condition (on S) for the displacement itself, instead of for the traction. For example, S might be rigid. It might seem at first that the traction on S and the displacement on S are independent properties of the displacement field throughout V. This is not so, however, and it is important for an intuitive understanding of Sections 2.3–2.5 to appreciate that traction over S determines the displacement over S, and vice versa.

2.3.1 UNIQUENESS THEOREM

The displacement $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ throughout the volume V with surface S is uniquely determined after time t_0 by the initial values of displacement and particle velocity at t_0 throughout V; and by values at all times $t \geq t_0$ of (i) the body forces $\mathbf{f}$ and the heat $\mathcal{Q}$ supplied throughout V; (ii) the tractions $\mathbf{T}$ over any part S_1 of S; and (iii) the displacement over the remainder S_2 of S, with $S_1 + S_2 = S$. (Either of S_1 or S_2 can be the whole of S.)

PROOF

Suppose $\mathbf{u}_1$ and $\mathbf{u}_2$ are any solutions for $\mathbf{u}$ that satisfy the same initial conditions and are set up by the same values for (i)–(iii). Then, using linearity, the difference $\mathbf{U} \equiv \mathbf{u}_1 - \mathbf{u}_2$ is a displacement field having zero initial conditions, and is set up by zero body forces, zero heating, zero traction on S_1, and $\mathbf{U} = \mathbf{0}$ on S_2. It remains to prove that $\mathbf{U} = \mathbf{0}$ throughout V for $t > t_0$.

From (2.22), the rate of doing mechanical work in the displacement field $\mathbf{U}$ is clearly zero throughout V and S_1 and S_2 for $t \geq t_0$. The last equality in (2.22) can be integrated from t_0 to t, and, together with the zero initial conditions and the use of a strain–energy function ($\mathbf{U}$ involves adiabatic changes), it follows that

$$\iiint_V \frac{1}{2}\rho \dot{U}_i \dot{U}_i \, dV + \iiint_V \frac{1}{2} c_{ijkl} U_{i,j} U_{k,l} \, dV = 0.$$

Both the kinetic and strain energies are positive definite, so that $\dot{U}_i = 0$ for $t \geq t_0$. But $U_i = 0$ at $t = t_0$, and hence $\mathbf{U} = \mathbf{0}$ throughout V for $t \geq t_0$.

2.3.2 RECIPROCITY THEOREMS

We shall state and prove several general relationships between a pair of solutions for the displacement throughout an elastic body V.

BOX **2.4**

Use of the term "homogeneous" as applied to equations and boundary conditions

The equation for elastic displacement is $\mathbf{L}(\mathbf{u}) = \mathbf{f}$, where $\mathbf{L}$ is the vector differential operator defined on the components of $\mathbf{u}$ by

$$(\mathbf{L}(\mathbf{u}))_i \equiv \rho \ddot{u}_i - (c_{ijkl} u_{k,l})_{,j}.$$

If body forces are absent, then the equation $\mathbf{L}(\mathbf{u}) = \mathbf{0}$ for $\mathbf{u}$ is said to be *homogeneous*. A *homogeneous boundary condition* on the surface S is one for which *either* the displacement *or* the traction vanishes at every point of the surface. If a solution to the homogeneous equation is multiplied by a constant, the result is still a solution (unlike the outcome of multiplying a solution to the inhomogeneous equation, $\mathbf{L}(\mathbf{u}) = \mathbf{f}$ with $\mathbf{f} \neq \mathbf{0}$, by a constant).

This terminology is reminiscent of linear algebra, for which a system of n equations in n unknowns, in the form $\mathbf{Ax} = \mathbf{0}$, is also said to be homogeneous. Here, $\mathbf{x}$ is a column vector and $\mathbf{A}$ is some $n \times n$ matrix. It is well known that nontrivial solutions ($\mathbf{x} \neq \mathbf{0}$) can exist, but only if $\mathbf{A}$ has a special property (namely, a zero determinant). The corresponding result in dynamic elasticity is that motions can occur throughout a finite elastic volume V without any body forces and with a homogeneous boundary condition over the surface of V. These are the *free oscillations* or *normal modes* of the body, which can occur only at certain frequencies. See Chapter 8.

Suppose that $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is one of these displacement fields, and that $\mathbf{u}$ is due to body forces $\mathbf{f}$ and boundary conditions on S and initial conditions at time $t = 0$. Let $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ be another displacement field due to body forces $\mathbf{g}$ and to boundary conditions and initial conditions (at $t = 0$) which in general are different from the conditions for $\mathbf{u}$. To distinguish the tractions on surfaces normal to $\mathbf{n}$ in these two cases, we shall use the notation $\mathbf{T}(\mathbf{u}, \mathbf{n})$ for the traction due to the displacement $\mathbf{u}$ and, similarly, $\mathbf{T}(\mathbf{v}, \mathbf{n})$ for the traction due to $\mathbf{v}$.

The first reciprocal relation to note between $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{aligned} &\iiint_V (\mathbf{f} - \rho\ddot{\mathbf{u}}) \cdot \mathbf{v}\, dV + \iint_S \mathbf{T}(\mathbf{u}, \mathbf{n}) \cdot \mathbf{v}\, dS \\ &= \iiint_V (\mathbf{g} - \rho\ddot{\mathbf{v}}) \cdot \mathbf{u}\, dV + \iint_S \mathbf{T}(\mathbf{v}, \mathbf{n}) \cdot \mathbf{u}\, dS. \end{aligned} \tag{2.34}$$

This result is due to Betti. It can easily be proved by substitution from (2.17) and (2.16) and then applying the divergence theorem to reduce the left side to $\iiint_V c_{ijkl} v_{i,j} u_{k,l}\, dV$. Similarly, the right-hand side reduces to $\iiint_V c_{ijkl} u_{i,j} v_{k,l}\, dV$, and (2.34) follows from the symmetry $c_{ijkl} = c_{klij}$.

Note that Betti's theorem does not involve initial conditions for $\mathbf{u}$ or $\mathbf{v}$. Furthermore, it remains true even if the quantities $\mathbf{u}, \ddot{\mathbf{u}}, \mathbf{T}(\mathbf{u}, \mathbf{n})$, and $\mathbf{f}$ are evaluated at time t_1 but $\mathbf{v}, \ddot{\mathbf{v}}, \mathbf{T}(\mathbf{v}, \mathbf{n})$, and $\mathbf{g}$ are evaluated at a different time t_2. If we choose $t_1 = t$ and $t_2 = \tau - t$

BOX **2.5**
Parallels

A rearrangement of Betti's relation (2.34) gives

$$\iiint_V \{v_i(c_{ijkl}u_{k,l})_{,j} - u_i(c_{ijkl}v_{k,l})_{,j}\}\, dV = \iint_S \{v_i T_i(\mathbf{u},\mathbf{n}) - u_i T_i(\mathbf{v},\mathbf{n})\}\, dS.$$

This is a vector theorem for the second-order spatial derivatives occurring in the wave equation of elasticity, which is analogous to Green's theorem

$$\iiint_V (\psi\nabla^2\phi - \phi\nabla^2\psi)\, dV = \iint_S \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right) dS$$

for scalars and the Laplacian operator. Green's theorem is a working tool for studying inhomogeneous equations, such as $\nabla^2\phi = -4\pi\rho$, and we shall use Betti's theorem for the elastic wave equation, in which the inhomogeneity is the body-force term.

There are many further analogies between Dirichlet problems (for potentials that are zero on S) and elasticity problems with rigid boundaries; and between Neumann problems ($\partial\phi/\partial n = 0$ on S) and traction-free boundaries.

and integrate (2.34) over the temporal range 0 to τ, then the acceleration terms reduce to terms that depend only on the initial and final values, since

$$\begin{aligned}\int_0^\tau \rho\{\ddot{\mathbf{u}}(t)\cdot\mathbf{v}(\tau-t) - \mathbf{u}(t)\cdot\ddot{\mathbf{v}}(\tau-t)\}\, dt \\ = \rho\int_0^\tau \frac{\partial}{\partial t}\{\dot{\mathbf{u}}(t)\cdot\mathbf{v}(\tau-t) + \mathbf{u}(t)\cdot\dot{\mathbf{v}}(\tau-t)\}\, dt \\ = \rho\,\{\dot{\mathbf{u}}(\tau)\cdot\mathbf{v}(0) - \dot{\mathbf{u}}(0)\cdot\mathbf{v}(\tau) + \mathbf{u}(\tau)\cdot\dot{\mathbf{v}}(0) - \mathbf{u}(0)\cdot\dot{\mathbf{v}}(\tau)\}\,.\end{aligned}$$

If there is some time τ_0 before which $\mathbf{u}$ and $\mathbf{v}$ are everywhere zero throughout V (and hence $\dot{\mathbf{u}} = \dot{\mathbf{v}} = 0$ for $\tau \le \tau_0$), then it follows that the convolution

$$\int_{-\infty}^{\infty} \rho\{\ddot{\mathbf{u}}(t)\cdot\mathbf{v}(\tau-t) - \mathbf{u}(t)\cdot\ddot{\mathbf{v}}(\tau-t)\}\, dt$$

is zero. We deduce from Betti's theorem the important result, for displacement fields with a quiescent past, that

$$\begin{aligned}\int_{-\infty}^{\infty} dt \iiint_V \{\mathbf{u}(\mathbf{x},t)\cdot\mathbf{g}(\mathbf{x},\tau-t) - \mathbf{v}(\mathbf{x},\tau-t)\cdot\mathbf{f}(\mathbf{x},t)\}\, dV \\ = \int_{-\infty}^{\infty} dt \iint_S \{\mathbf{v}(\mathbf{x},\tau-t)\cdot\mathbf{T}(\mathbf{u}(\mathbf{x},t),\mathbf{n}) - \mathbf{u}(\mathbf{x},t)\cdot\mathbf{T}(\mathbf{v}(\mathbf{x},\tau-t),\mathbf{n})\}\, dS.\end{aligned} \tag{2.35}$$

2.4 Introducing Green's Function for Elastodynamics

A major aim of this chapter and the next is the development of a representation for the displacements that typically occur in seismology. The representation will be a formula for the displacement (at a general point in space and time) in terms of the quantities that originated the motion, and we have seen (in the uniqueness theorem) that these are body forces and applied tractions or displacements over the surface of the elastic body under discussion. For earthquake faulting, the seismic source is complicated in that it extends over a finite fault plane (or a finite volume) and over a finite amount of time, and in general involves motions (at the source) that have varying direction and magnitude. We shall find that the representation theorem is really nothing but a bookkeeping device by which the displacement from realistic source models is synthesized from the displacement produced by the simplest of sources—namely, the unidirectional unit impulse, which is localized precisely in both space and time.

The displacement field from such a simple source is the elastodynamic Green function. If the unit impulse is applied at $\mathbf{x} = \xi$ and $t = \tau$ and in the n-direction (see (2.4), taking A = unit constant with dimensions of impulse), then we denote the ith component of displacement at general $(\mathbf{x}, t)$ by $G_{in}(\mathbf{x}, t; \xi, \tau)$. Clearly, this Green function is a tensor (we shall work throughout with Cartesian tensors, and therefore do not distinguish between tensors and dyadics). It depends on both receiver and source coordinates, and satisfies the equation

$$\rho \frac{\partial^2}{\partial t^2} G_{in} = \delta_{in}\, \delta(\mathbf{x} - \xi)\, \delta(t - \tau) + \frac{\partial}{\partial x_j}\left(c_{ijkl} \frac{\partial}{\partial x_l} G_{kn} \right) \qquad (2.36)$$

throughout V. We shall invariably use the initial conditions that $\mathbf{G}(\mathbf{x}, t; \xi, \tau)$ and $\partial\{\mathbf{G}(\mathbf{x}, t; \xi, \tau)\}/\partial t$ are zero for $t \leq \tau$ and $\mathbf{x} \neq \xi$. To specify $\mathbf{G}$ uniquely it remains to state the boundary conditions on S, and we shall use a variety of different boundary conditions in different applications.

If the boundary conditions are independent of time (e.g., S always rigid), then the time origin can be shifted at will, and we see from (2.36) that $\mathbf{G}$ depends on t and τ only via the combination $t - \tau$. Hence

$$\mathbf{G}(\mathbf{x}, t; \xi, \tau) = \mathbf{G}(\mathbf{x}, t - \tau; \xi, 0) = \mathbf{G}(\mathbf{x}, -\tau; \xi, -t), \qquad (2.37)$$

which is a reciprocal relation for source and receiver times.

If $\mathbf{G}$ satisfies homogeneous boundary conditions on S, then (2.35) can be used to obtain an important reciprocal relation for source and receiver positions. One takes $\mathbf{f}$ to be a unit impulse applied in the m-direction at $\mathbf{x} = \xi_1$ and time $t = \tau_1$, and $\mathbf{g}$ to be a unit impulse applied in the n-direction at $\mathbf{x} = \xi_2$ and time $t = -\tau_2$. Then $u_i = G_{im}(\mathbf{x}, t; \xi_1, \tau_1)$ and $v_i = G_{in}(\mathbf{x}, t; \xi_2, -\tau_2)$, so that (2.35) directly yields

$$G_{nm}(\xi_2, \tau + \tau_2; \xi_1, \tau_1) = G_{mn}(\xi_1, \tau - \tau_1; \xi_2, -\tau_2). \qquad (2.38)$$

Choosing $\tau_1 = \tau_2 = 0$, this becomes

$$G_{nm}(\xi_2, \tau; \xi_1, 0) = G_{mn}(\xi_1, \tau; \xi_2, 0), \tag{2.39}$$

which specifies a purely spatial reciprocity. Choosing $\tau = 0$ in (2.38) gives

$$G_{nm}(\xi_2, \tau_2; \xi_1, \tau_1) = G_{mn}(\xi_1, -\tau_1; \xi_2, -\tau_2), \tag{2.40}$$

which specifies a space–time reciprocity.

The actual computation of an elastodynamic Green function can itself be a complicated problem. We shall take up this subject in later chapters, beginning in Chapter 4 with the simplest of elastic solids (homogeneous, isotropic, infinite) and moving on to the case of large separation between source and receiver in inhomogeneous media.

2.5 Representation Theorems

If the integrated form of Betti's theorem, our equation (2.35), is used with a Green function for one of the displacement fields, then a representation for the other displacement field becomes available.

Specifically, suppose we are interested in finding an expression for the displacement **u** due both to body forces **f** throughout V and to boundary conditions on S. We substitute into (2.35) the body force $g_i(\mathbf{x}, t) = \delta_{in}\,\delta(\mathbf{x} - \xi)\,\delta(t)$, for which the corresponding solution is $v_i(\mathbf{x}, t) = G_{in}(\mathbf{x}, t; \xi, 0)$, and find

$$\begin{aligned} u_n(\xi, \tau) = &\int_{-\infty}^{\infty} dt \iiint_V f_i(\mathbf{x}, t) G_{in}(\mathbf{x}, \tau - t; \xi, 0)\, dV \\ &+ \int_{-\infty}^{\infty} dt \iint_S \{G_{in}(\mathbf{x}, \tau - t; \xi, 0) T_i(\mathbf{u}(\mathbf{x}, t), \mathbf{n}) \\ &\qquad - u_i(\mathbf{x}, t) c_{ijkl} n_j G_{kn,l}(\mathbf{x}, \tau - t; \xi, 0)\}\, dS. \end{aligned}$$

Before giving a physical interpretation of this equation, it is helpful to interchange the symbols **x** and ξ and the symbols t and τ. This permits $(\mathbf{x}, t)$ to be the general position and time at which a displacement is to be evaluated, regarded as an integral over volume and surface elements at varying ξ with a temporal convolution. The result is

$$\begin{aligned} u_n(\mathbf{x}, t) = &\int_{-\infty}^{\infty} d\tau \iiint_V f_i(\xi, \tau) G_{in}(\xi, t - \tau; \mathbf{x}, 0)\, dV(\xi) \\ &+ \int_{-\infty}^{\infty} d\tau \iint_S \{G_{in}(\xi, t - \tau; \mathbf{x}, 0) T_i(\mathbf{u}(\xi, \tau), \mathbf{n}) \\ &\qquad - u_i(\xi, \tau) c_{ijkl} n_j G_{kn,l}(\xi, t - \tau; \mathbf{x}, 0)\}\, dS(\xi). \end{aligned} \tag{2.41}$$

This is our first representation theorem. It states a way in which the displacement **u** at a certain point is made up from contributions due to the force **f** throughout V, plus contributions due to the traction $\mathbf{T}(\mathbf{u}, \mathbf{n})$ and to the displacement $\mathbf{u}$ itself on S. However, the way in which each of these three contributions is weighted is unsatisfactory, since each involves a Green function with source at $\mathbf{x}$ and observation point at ξ. (Note that the last term in (2.41) involves differentiation with respect to ξ_l.) We want $\mathbf{x}$ to be the observation point, so that the total displacement obtained there can be regarded as the sum (integral) of contributing displacements at $\mathbf{x}$ due to each volume element and surface element. The reciprocal theorem for $\mathbf{G}$ must be invoked, but this will require extra conditions on Green's function itself, since the equation $G_{in}(\xi, t-\tau; \mathbf{x}, 0) = G_{ni}(\mathbf{x}, t-\tau; \xi, 0)$ (see (2.39)) was proved only if $\mathbf{G}$ satisfies homogeneous boundary conditions on S, whereas (2.41) is valid for *any* Green function set up by an impulsive force in the n-direction at $\xi = \mathbf{x}$ and $\tau = t$.

We shall examine two different cases. Suppose, first, that Green's function is determined with S as a rigid boundary. We write $\mathbf{G}^{\text{rigid}}$ for this function and $G_{in}^{\text{rigid}}(\xi, t-\tau; \mathbf{x}, 0) = 0$ for ξ in S. Then (2.41) becomes

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} dt \iiint_V f_i(\xi, \tau) G_{ni}^{\text{rigid}}(\mathbf{x}, t-\tau; \xi, 0)\, dV$$
$$- \int_{-\infty}^{\infty} dt \iint_S u_i(\xi, \tau) c_{ijkl} n_j \frac{\partial}{\partial \xi_l} G_{nk}^{\text{rigid}}(\mathbf{x}, t-\tau; \xi, 0)\}\, dS. \tag{2.42}$$

Alternatively, we can use $\mathbf{G}^{\text{free}}$ as Green's function, so that the traction $c_{ijkl} n_j (\partial/\partial \xi_l) G_{kn}^{\text{free}}(\xi, t-\tau; \mathbf{x}, 0)$ is zero for ξ in S, finding

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} dt \iiint_V f_i(\xi, \tau) G_{in}^{\text{free}}(\mathbf{x}, t-\tau; \xi, 0)\, dV$$
$$+ \int_{-\infty}^{\infty} dt \iint_S G_{ni}^{\text{free}}(\mathbf{x}, t-\tau; \xi, 0) T_i(\mathbf{u}(\xi, \tau), \mathbf{n})\, dS. \tag{2.43}$$

Equations (2.41)–(2.43) are all different forms of the representation theorem and each has its special uses. Taken together, they seem to imply a contradiction to the question of whether $\mathbf{u}(\mathbf{x}, t)$ depends upon displacement on S (see (2.42)) or traction (see (2.43)) or both (see (2.41)). But since traction and displacement cannot be specified independently on the surface of an elastic medium, there is no contradiction. In (2.41), the Green function is not completely defined.

The surface on which values of traction (or displacement) are explicitly required has been taken, in this chapter, as external to the volume V. It is often useful instead to take this surface to include two adjacent internal surfaces, being the opposite faces of a buried fault. Specialized forms of the representation theorem can then be developed, which enable one to analyze the earthquakes set up by activity on a buried fault. This subject is central to earthquake source theory, taken up in Chapter 3 and developed much further in Chapters 10 and 11.

So far, we have considered only Cartesian coordinate systems. In practice, the seismologist is often required to use non-Cartesian coordinates that allow the physical relationship between components of displacement, stress, and strain to be simplified for the geometry of a particular problem. We do this because it is often found that a boundary condition must be applied on a surface on which a general curvilinear coordinate is constant. Many texts derive formulas in general orthogonal coordinates for vector operations such as grad, div, curl, and ∇^2, but rather more is needed to analyze the vector operations required in elasticity, as we next discuss.

2.6 Strain–Displacement Relations and Displacement–Stress Relations in General Orthogonal Curvilinear Coordinates

Continuing with the notation developed in Box 2.6, we shall first obtain relations between strain components e^{pq} and displacement components u^r that generalize the usual Cartesian result $e_{ij} = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$. By e^{pq}, we merely mean the components of the Cartesian second-order tensor $\mathbf{e}$, referred to rotated Cartesian axes, which are defined (at the point of interest) to lie along the directions $\mathbf{n}^1, \mathbf{n}^2, \mathbf{n}^3$. Thus we emphasize the physical components of strain, rather than the general tensor components (which may not even have the dimensions of strain). Our problem is to express e^{pq} in terms of derivatives (with respect to c^1, c^2, c^3) of the physical components of displacement also resolved along $\mathbf{n}^1, \mathbf{n}^2, \mathbf{n}^3$: the difficulties that arise are due (a) to spatial changes in the scaling functions h^1, h^2, h^3, and (b) to spatial changes in the directions $\mathbf{n}^1, \mathbf{n}^2, \mathbf{n}^3$.

Direction cosines of the rotated Cartesian axis along $\mathbf{n}^p$ are (n_1^p, n_2^p, n_3^p), referred to the Cartesian axes $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3$ (which are in the same fixed direction at every point). Therefore, from the fundamental transformation property of Cartesian vector and tensor components,

$$u^p = n_i^p u_i \quad \text{(summation is retained for repeated subscripts)} \tag{2.44}$$

$$\begin{aligned}
e^{pq} &= n_i^p n_j^q e_{ij} \\
&= \frac{1}{h^p h^q}\frac{\partial x_i}{\partial c^p}\frac{\partial x_j}{\partial c^q}\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{(from (3) in Box 2.6; no summation over superscripts)} \\
&= \frac{1}{2h^p h^q}\left(\frac{\partial x_i}{\partial c^p}\frac{\partial u_i}{\partial c^q} + \frac{\partial x_i}{\partial c^q}\frac{\partial u_i}{\partial c^p}\right) \quad \text{(reversing the chain rule in the previous line)} \\
&= \frac{1}{2h^q}\left[\frac{\partial}{\partial c^q}\left(\frac{u_i}{h^p}\frac{\partial x_i}{\partial c^p}\right) - u_i\frac{\partial}{\partial c^q}\left(\frac{1}{h^p}\frac{\partial x_i}{\partial c^p}\right)\right] \\
&\quad + \frac{1}{2h^p}\left[\frac{\partial}{\partial c^p}\left(\frac{u_i}{h^q}\frac{\partial x_i}{\partial c^q}\right) - u_i\frac{\partial}{\partial c^p}\left(\frac{1}{h^q}\frac{\partial x_i}{\partial c^q}\right)\right] \\
&= \frac{1}{2h^q}\frac{\partial u^p}{\partial c^q} + \frac{1}{2h^p}\frac{\partial u^q}{\partial c^p} - \frac{1}{2}u_i\left[\frac{1}{h^q}\frac{\partial}{\partial c^q}n_i^p + \frac{1}{h^p}\frac{\partial}{\partial c^p}n_i^q\right] \quad \text{(from (3) in Box 2.6, and (2.44))} \\
&= \frac{1}{2h^q}\frac{\partial u^p}{\partial c^q} + \frac{1}{2h^p}\frac{\partial u^q}{\partial c^p} - \frac{1}{2}\mathbf{u}\cdot\left[\frac{1}{h^q}\frac{\partial \mathbf{n}^p}{\partial c^q} + \frac{1}{h^p}\frac{\partial \mathbf{n}^q}{\partial c^p}\right].
\end{aligned}$$

BOX **2.6**

General properties of orthogonal curvilinear coordinates

Consider a point at the vector position $\mathbf{x}$ to be specified by three parameters, c^1, c^2, c^3. That is, each of the three components of $\mathbf{x}$ (in some Cartesian coordinate system) is a scalar function of the c^p:

$$x_i = x_i(c^1, c^2, c^3) \qquad (i = 1, 2, 3).$$

We suppose that these functions x_i have continuous derivatives and that there are inverse functions

$$c^p = c^p(x_1, x_2, x_3) \qquad (p = 1, 2, 3) \qquad \text{or} \qquad c^p = c^p(\mathbf{x}),$$

so that the equation $c^p =$ constant can be thought of as a coordinate surface for each p, and these three surfaces intersect in pairs on lines along which only one of the c^1, c^2, c^3 is varying. We use superscripts for quantities identified with the general curvilinear system.

Let $\mathbf{n}^p$ be the unit normal to the coordinate surface $c^p =$ constant, and suppose $\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$ both lie in this surface. Then $c^p(\mathbf{x}) = c^p(\mathbf{x} + d\mathbf{x})$, and hence $d\mathbf{x} \cdot \nabla c^p = 0$, using the Taylor expansion of $c^p(\mathbf{x} + d\mathbf{x})$. Since $d\mathbf{x}$ is *any* line element within the surface, it follows that ∇c^p is normal to $c^p =$ constant, and ∇c^p must be parallel to $\mathbf{n}^p$.

Let the length of vector ∇c^p be $1/h^p$ (a scaling factor). Then

$$\mathbf{n}^p = h^p \nabla c^p. \tag{1}$$

(We drop the summation convention for superscripts, but retain it for subscripts, these being related to the original Cartesian system.)

We shall assume that c^1, c^2, c^3 form a *right-handed orthogonal system*, i.e., that

$$\mathbf{n}^p \cdot \mathbf{n}^q = \delta^{pq} \qquad \text{(the Kronecker delta)}, \tag{2}$$

and that $\mathbf{n}^3 = \mathbf{n}^1 \times \mathbf{n}^2$.

Using n_i^p for the ith Cartesian component of $\mathbf{n}^p$, we can now obtain an important relation between $\mathbf{n}^p$ and $\partial\mathbf{x}/\partial c^p$, as follows:

$$\mathbf{n}^p = n_i^p \hat{\mathbf{x}}_i = n_i^p \frac{\partial \mathbf{x}}{\partial x_i} = \sum_q n_i^p \frac{\partial \mathbf{x}}{\partial c^q}\frac{\partial c^q}{\partial x_i} \qquad \text{(the chain rule)}$$

$$= \sum_q n_i^p \frac{n_i^q}{h^q}\frac{\partial \mathbf{x}}{\partial c^q} \qquad \text{(from (1))} \qquad = \sum_q \frac{\delta^{pq}}{h^q}\frac{\partial \mathbf{x}}{\partial c^q} \qquad \text{(from (2))}$$

and hence

$$\mathbf{n}^p = \frac{1}{h^p}\frac{\partial \mathbf{x}}{\partial c^p}. \tag{3}$$

A small change $d\mathbf{x}$ in position is associated with a small change in each of coordinates c^1, c^2, c^3 by $d\mathbf{x} = \sum_p (\partial\mathbf{x}/\partial c^p) dc^p$, and the magnitude of this change is given by

$$(ds)^2 = d\mathbf{x} \cdot d\mathbf{x} = \sum_p \frac{\partial \mathbf{x}}{\partial c^p} dc^p \cdot \sum_q \frac{\partial \mathbf{x}}{\partial c^q} dc^q$$

$$= (h^1\, dc^1)^2 + (h^2\, dc^2)^2 + (h^3\, dc^3)^2 \qquad \text{(from (3) and (2))}. \tag{4}$$

(continued)

BOX **2.6** *(continued)*

This result leads to one of the quickest ways of actually finding the scaling functions: the Euclidean distance associated with increment dc^1 along $\mathbf{n}^1$ is $h^1 dc^1$; and similarly for h^2 and h^3.

In Section 2.6, we need formulas for derivatives of the type $\partial \mathbf{n}^p / \partial c^q$ in terms of the undifferentiated normals. From (2) and (3), the equations to be satisfied are

$$\mathbf{n}^p \cdot \frac{\partial \mathbf{n}^q}{\partial c^r} + \mathbf{n}^q \cdot \frac{\partial \mathbf{n}^p}{\partial c^r} = 0 \qquad \text{(18 different scalar equations)}$$
$$\frac{\partial}{\partial c^q}(h^p \mathbf{n}^p) = \frac{\partial}{\partial c^p}(h^q \mathbf{n}^q) \qquad \text{(3 nontrivial vector equations).} \tag{5}$$

The above are 27 different scalar equations for the 27 scalar unknowns in $\partial \mathbf{n}^p / \partial c^q$, and hence are exactly enough to determine the solution. In vector form, this solution is

$$\frac{\partial \mathbf{n}^p}{\partial c^q} = \frac{\mathbf{n}^q}{h^p} \frac{\partial h^q}{\partial c^p} - \delta^{pq} \left[\frac{\mathbf{n}^1}{h^1} \frac{\partial h^p}{\partial c^1} + \frac{\mathbf{n}^2}{h^2} \frac{\partial h^p}{\partial c^2} + \frac{\mathbf{n}^3}{h^3} \frac{\partial h^p}{\partial c^3} \right], \tag{6}$$

as may be verified by direct substitution back into (5).

In this form, we can use the final equation of Box 2.6 to obtain

$$e^{pq} = \frac{1}{2} \left[\frac{h^p}{h^q} \frac{\partial}{\partial c^q} \left(\frac{u^p}{h^p} \right) + \frac{h^q}{h^p} \frac{\partial}{\partial c^p} \left(\frac{u^q}{h^q} \right) \right] + \frac{\delta^{pq}}{h^q} \left[\frac{u^1}{h^1} \frac{\partial h^p}{\partial c^1} + \frac{u^2}{h^2} \frac{\partial h^p}{\partial c^2} + \frac{u^3}{h^3} \frac{\partial h^p}{\partial c^3} \right], \tag{2.45}$$

in which all reference to the Cartesian system (x_1, x_2, x_3) has at last been eliminated. Only the first square bracket is required for the off-diagonal components $(p \neq q)$, but for a typical diagonal component (2.45) reduces to, e.g.,

$$e^{11} = \frac{1}{h^1} \frac{\partial u^1}{\partial c^1} + \frac{u^2}{h^1 h^2} \frac{\partial h^1}{\partial c^2} + \frac{u^3}{h^3 h^1} \frac{\partial h^1}{\partial c^3}. \tag{2.46}$$

To obtain the displacement–stress relations for general orthogonal components of $\mathbf{u}$ and $\boldsymbol{\tau}$, we follow steps similar to the derivation of $\rho \ddot{u}_i = \tau_{ij,j}$ given in Section 2.1 for fixed Cartesian directions. The principal difficulty lies in interpreting $\iint_S \mathbf{T}\, dS$, the integral of traction acting across the surface S with volume V. With $\boldsymbol{\nu}$ as the outward normal on dS,

$$\begin{aligned} T_i(\boldsymbol{\nu})\, dS &= \tau_{ij} \nu_j \, dS \\ &= \sum_{p,q} \tau^{pq} n_i^p n_j^q \nu_j dS \qquad \text{(transformation to components in rotated Cartesians)} \\ &= \sum_{p,q} \tau^{pq} n_i^p \nu^q \, dS, \end{aligned}$$

where ν^q is the component of the normal to dS, resolved along $\mathbf{n}^q$.

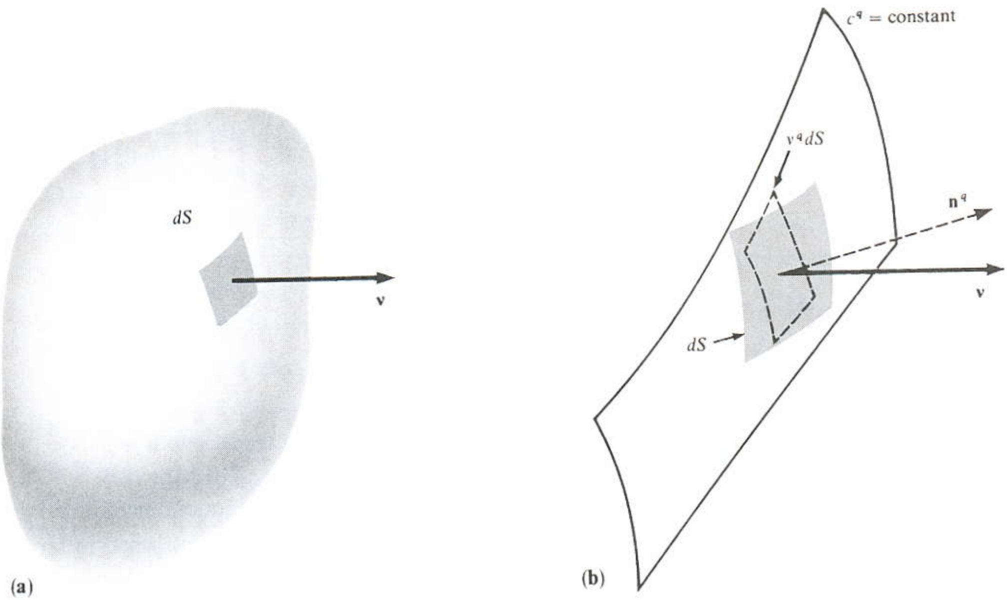

FIGURE 2.6
The projection of dS onto the surface c^q = constant. The resulting area on the coordinate surface is $\nu^q\,dS$. (**a**) Shown here is dS as part of the surface of V. (**b**) The projection of dS, in broken outline, onto the coordinate surface c^q = constant.

Now $\nu^q\,dS$ is the projection of dS onto the surface $c^q = \text{constant}$ (see Fig. 2.6), so that $\nu^1\,dS = h^2h^3\,dc^2\,dc^3$; similarly for $\nu^2\,dS$ and $\nu^3\,dS$. It follows that

$$\begin{aligned}\iint_S T_i\,dS &= \sum_p \iint_S [\tau^{p1}n_i^p h^2h^3\,dc^2\,dc^3 + \tau^{p2}n_i^p h^3h^1\,dc^3\,dc^1 \\ &\qquad + \tau^{p3}n_i^p h^1h^2\,dc^1\,dc^2] \\ &= \sum_p \iiint_V \left[\frac{\partial}{\partial c^1}(\tau^{p1}n_i^p h^2h^3) + \frac{\partial}{\partial c^2}(\tau^{p2}n_i^p h^3h^1) \right. \\ &\qquad \left. + \frac{\partial}{\partial c^3}(\tau^{p3}n_i^p h^1h^2)\right] dc^1\,dc^2\,dc^3.\end{aligned}$$

But the physical volume element dV is $h^1h^2h^3\,dc^1\,dc^2\,dc^3$, so from steps parallel to the derivation of the equation of motion given in Section 2.1, we find here that

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \mathbf{f} + \frac{1}{h^1h^2h^3}\sum_{p,q}\frac{\partial}{\partial c^q}\left(\tau^{pq}\,\mathbf{n}^p\,\frac{h^1h^2h^3}{h^q}\right). \tag{2.47}$$

Again, the derivative $\partial \mathbf{n}^p / \partial c^q$ is needed (see (6) in Box 2.6), and by resolving (2.47) along direction $\mathbf{n}^1$ we find

$$\rho \frac{\partial^2 u^1}{\partial t^2} = f^1 + \frac{1}{h^1 h^2 h^3} \left[\frac{\partial}{\partial c^1}(\tau^{11} h^2 h^3) + \frac{\partial}{\partial c^2}(\tau^{12} h^3 h^1) + \frac{\partial}{\partial c^3}(\tau^{31} h^1 h^2) \right] + \frac{\tau^{12}}{h^1 h^2} \frac{\partial h^1}{\partial c^2} + \frac{\tau^{31}}{h^3 h^1} \frac{\partial h^1}{\partial c^3} - \frac{\tau^{22}}{h^1 h^2} \frac{\partial h^2}{\partial c^1} - \frac{\tau^{33}}{h^3 h^1} \frac{\partial h^3}{\partial c^1}. \tag{2.48}$$

Similar results for $\rho \ddot{u}^2$ and $\rho \ddot{u}^3$ can be found from a permutation of superscripts in (2.48).

The stress–strain relation, $\tau_{ij} = c_{ijkl} e_{kl}$, becomes

$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} \tag{2.49}$$

in isotropic media. We have used (2.33) here: λ and μ are (in general) functions of position, and $e_{kk} = e_{11} + e_{22} + e_{33}$ is the volumetric strain. Equation (2.49) is expressed in terms of components in the fixed-direction Cartesian system, but the corresponding result for physical components in the general orthogonal system has the same form. It is

$$\tau^{pq} = \lambda \delta^{pq} \sum_r e^{rr} + 2\mu e^{pq}, \tag{2.50}$$

since isotropy of the medium implies $c^{pqrs} = c_{pqrs}$, and we can again use (2.33). The only difference in the form of (2.49) and (2.50) is due to our using a summation convention for subscripts but not for superscripts.

Applications of (2.46), (2.48), and (2.50) are common in spherical polars (r, θ, ϕ), for which the scaling functions h^1, h^2, h^3 become, respectively, $1, r, r \sin\theta$; and, in cylindrical polars, (r, ϕ, z) with scaling functions $1, r, 1$. In Chapter 4 we shall use orthogonal curvilinear coordinates associated with the wavefronts and rays that emanate from a point source in an inhomogeneous isotropic medium. Our convention of superscripts is convenient for the derivation of (2.45)–(2.50), but in applications the superscripts are usually replaced by subscripts that directly indicate the coordinate of interest. Thus, if (c^1, c^2, c^3) are the spherical polars (r, θ, ϕ), one refers to e^{12} as $e_{r\theta}$, to u^3 as u_ϕ, and to $\mathbf{n}^2$ as $\hat{\boldsymbol{\theta}}$.

Suggestions for Further Reading

Achenbach, J. D. *Wave Propagation in Elastic Solids*. Amsterdam: North-Holland, 1973.

Fung, Y. C. *Foundations of Solid Mechanics*. Englewood Cliffs, New Jersey: Prentice-Hall, 1965.

Jeffreys, H. *Cartesian Tensors*. Cambridge University Press, 1965.

Love, A. E. H. *A Treatise on the Mathematical Theory of Elasticity*. New York: Dover Publications, 1944.

Malvern, L. E. *Introduction to the Mechanics of a Continuous Medium*. Englewood Cliffs, New Jersey: Prentice-Hall, 1969.

Problems

2.1 Show that the displacement equation for infinitesimal motion in an elastic anisotropic medium is

$$\rho \ddot{u}_i = f_i + (c_{ijkl} u_{k,l})_{,j}.$$

If the medium is homogeneous and isotropic, show that this displacement equation becomes

$$\rho \ddot{u}_i = f_i + (\lambda + \mu) u_{j,ji} + \mu u_{i,jj}.$$

The above two equations are the ith Cartesian component of a vector equation. Show that this vector equation, for the homogeneous isotropic medium, is

$$\rho \ddot{\mathbf{u}} = \mathbf{f} + (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u}).$$

2.2 From the expression for $\varepsilon_{ijk}\varepsilon_{lmn}$ in Box 2.2, show that

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \quad \text{and} \quad \varepsilon_{ijk}\varepsilon_{jlm} = \delta_{im}\delta_{kl} - \delta_{il}\delta_{km}.$$

2.3 For an isotropic elastic solid in which the stress–strain relation is $\tau_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$, show that the strain–stress relation is

$$2\mu e_{ij} = -\frac{\lambda}{3\lambda + 2\mu}\tau_{kk}\delta_{ij} + \tau_{ij}.$$

2.4 What happens to the stress in a body if temperature is raised at fixed strain? Does the stress obey Hooke's law (2.18) or must this be modified in some way? (Recall that seismological applications of (2.18) are usually for adiabatic loading.)

2.5 We have shown how the displacement field $\mathbf{u}(\mathbf{x}, t)$ for an elastic body is given uniquely (e.g., by applied body forces and tractions). Show that body forces and tractions are given uniquely once $\mathbf{u}(\mathbf{x}, t)$ is known everywhere. (A proof "by construction" is very quick and simple.)

2.6 Do the relations (2.21)–(2.25) change if stress depends on strain rate (e.g., for a viscous medium)?

2.7 Obtain the traction due to displacement field $\mathbf{u}$ acting on area elements normal to $\mathbf{n}$, in the form

$$\mathbf{T}(\mathbf{u}, \mathbf{n}) = \lambda(\nabla \cdot \mathbf{u})\mathbf{n} + \mu\left(2\frac{\partial \mathbf{u}}{\partial n} + \mathbf{n} \times (\nabla \times \mathbf{u})\right).$$

Here

$$\frac{\partial \mathbf{u}}{\partial n} = (\mathbf{n} \cdot \nabla)\mathbf{u}.$$

2.8 The traction $\mathbf{T}$ in the previous question is a function of position $\mathbf{x}$, in the sense that $\mathbf{T} = \mathbf{T}(\mathbf{u}(\mathbf{x}), \mathbf{n})$.

a) Modify our derivation of (2.7) to show that traction is a continuous function of position, in the sense that

$$\mathbf{T}(\mathbf{x} + \delta\mathbf{x}) - \mathbf{T}(\mathbf{x}) \to \mathbf{0} \quad \text{as} \quad \delta\mathbf{x} \to \mathbf{0},$$

provided $\delta\mathbf{x}$ is taken parallel to the direction $\mathbf{n}$ that defines the orientation of area elements on which traction is evaluated.

b) Consider a book resting on a flat table. Is it true that traction is a continuous function of position on the surface of the table?

c) Check that your answers to a) and b) are not in conflict.

d) Show that $\tau_{yz}, \tau_{zx}, \tau_{zz}$ are continuous functions of z in any medium, but that τ_{zz} need not be continuous in the x- or y-directions; and that τ_{xx}, τ_{yy}, and τ_{xy} need not be continuous in the z-direction.

2.9 For a point at pressure P in a fluid, the stress tensor is isotropic and has components $\tau_{ij} = -P\delta_{ij}$. To emphasize the differences between stresses that are possible in a solid and those that are present in a fluid, it is convenient to define *deviatoric stresses* τ'_{ij} by $\tau_{ij} = \frac{1}{3}\tau_{kk}\delta_{ij} + \tau'_{ij}$ and *deviatoric strains* by $e_{ij} = \frac{1}{3}e_{kk}\delta_{ij} + e'_{ij}$. Show then that the strain energy $\mathcal{U}$ in an isotropic elastic medium is given by

$$\mathcal{U} = \tfrac{1}{2}[(\lambda + \tfrac{2}{3}\mu)e_{ii}e_{kk} + 2\mu e'_{ij}e'_{ij}].$$

Show that e_{ii} is the change in volume per unit volume (i.e., the volumetric strain). Hence $\mathcal{U}$ can be regarded as a sum of dilatational energy, $\frac{1}{2}(\lambda + \frac{2}{3}\mu)e_{ii}e_{kk}$, and shear strain energy $\mu e'_{ij}e'_{ij}$. Why must $\lambda + \frac{2}{3}\mu$ (often called the *bulk modulus*, denoted by κ) and μ be positive? Is it natural to call κ the *compressibility* or the *incompressibility*?

2.10 Consider two points, $\mathbf{x}$ and $\boldsymbol{\xi}$, in an elastic medium, and let the unit vectors $\mathbf{n}$ and $\boldsymbol{\nu}$ specify particular directions at $\mathbf{x}$ and $\boldsymbol{\xi}$, respectively. Show first that a unit impulse in the $\boldsymbol{\nu}$ direction at $\boldsymbol{\xi}$ leads to a displacement at $\mathbf{x}$ whose component in the $\mathbf{n}$ direction is given by $n_i G_{ip}(\mathbf{x}, t; \boldsymbol{\xi}, 0)\nu_p$. Then show that this displacement component equals the displacement component in the $\boldsymbol{\nu}$ direction at $\boldsymbol{\xi}$ caused by a unit impulse in the $\mathbf{n}$ direction at $\mathbf{x}$. (This result generalizes the reciprocity result given in (2.39), which was for an impulse taken along one of the coordinate axes and a displacement component also along a coordinate axis. The reciprocity is true for arbitrary directions $\mathbf{n}$ and $\boldsymbol{\nu}$.)